Sheaves of Lie Algebras of Vector Fields

Bas Janssens and Ori Yudilevich

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1 Truncated Lie algebras

Fifth lecture on Singer and Sternberg's 1965 paper [1], by Bas Janssens.

1.1 Introduction

Let V be a finite dimensional vector space. On the bosonic Fock space $S(V)$, we define the *creation operator* for $v \in V$ by $a_B^*(v)\colon u \mapsto v \vee u$ and the *annihilation operator* for $\alpha \in V^*$ by $a_B(\alpha)\colon u \mapsto i_\alpha u$ (contraction with α). Similarly, we define creation and annihilation operators on the fermionic Fock space $\bigwedge V$ by $a_F^*(v)\colon w \mapsto v \wedge w$ and $a_F(\alpha)\colon w \mapsto i_\alpha w$. On the bigraded module $S(V) \otimes \bigwedge(V)$, we consider the differential

$$\partial^* = \sum_i a_B^*(e_i) \otimes a_F(\epsilon^i)$$

where $(e_1, \ldots, e_n)$ is a basis of V and $(\epsilon^1, \ldots, \epsilon^n)$ the dual basis of V^*. Explicitly, it satisfies

$$\partial^*(u_1 \vee \ldots \vee u_k) \otimes (v_1 \wedge \ldots \wedge v_l) = \sum_{j=1}^{l} (-1)^j (v_j \vee u_1 \vee \ldots \vee u_k) \otimes (v_1 \wedge \ldots \hat{v}_j \ldots \wedge v_l).$$

Using the $[a_B(\alpha), a_B^*(v)] = \alpha(v)\mathbf{1}$ (CCR) and $\{a_F(\alpha), a_F^*(v)\} = \alpha(v)\mathbf{1}$ (CAR), one verifies the following commutation relations.

Proposition 1.1. *We have* $[\partial^*, a_B^*(v)] = 0$ *and* $\{\partial^*, a_F^*(v)\} = a_B^*(v)$.

Let W be a finite dimensional vector space. We continue to denote the extension of ∂^* to $S(V) \otimes W^* \otimes \bigwedge V$ by ∂^* and we denote its dual on $\overline{S}(V^*) \otimes W \otimes \bigwedge V^*$ by ∂. Let $\mathfrak{g} \subseteq \overline{S}(V^*) \otimes W$ be a graded subspace such that $\mathrm{ann}(\mathfrak{g}) \subseteq S(V) \otimes W^*$ is a $S(V)$-module. We have seen before that this is equivalent to $\mathfrak{g}_{k+1} \subseteq \mathrm{prol}(\mathfrak{g}_k)$ for all k. In this case, the operator ∂ restricts to $\mathfrak{g} \otimes \bigwedge V \subseteq S(V^*) \otimes W \otimes \bigwedge V$. Its homology is called the *Spencer homology* of $\mathfrak{g}$.

Theorem 1.2. *The Spencer homology is finite dimensional.*

Proof. The Spencer homology is dual to the cohomology of the operator ∂^* on $\mathfrak{g}^* \otimes \bigwedge V$. Since $\mathrm{ann}(\mathfrak{g})$ is an $S(V)$-module, the quotient $\mathfrak{g}^* = S(V) \otimes W^*/\mathrm{ann}(\mathfrak{g})$ is a finitely generated $S(V)$-module. Since $[\partial^*, a_B^*(v)] = 0$, the kernel $\mathrm{Ker}(\partial^*) \subseteq \mathfrak{g}^* \otimes \bigwedge V$ is an $S(V)$-submodule of a finitely generated $S(V)$-module, hence finitely generated itself by the Hilbert Basis Theorem. But $a_B^*(V)\mathrm{Ker}(\partial^*) \subseteq \mathrm{Im}(\partial^*)$ because $\{\partial^*, a_F^*(v)\} = a_B^*(v)$, so $\mathrm{Ker}(\partial^*)/\mathrm{Im}(\partial^*)$ is not only finitely generated, but even finite dimensional. $\square$

References

[1] I. M. Singer and Shlomo Sternberg. The infinite groups of Lie and Cartan. I. The transitive groups. *J. Analyse Math.*, 15:1–114, 1965.