Sheaves of Lie Algebras of Vector Fields

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1 Cartan's first fundamental theorem.

Second lecture on Singer and Sternberg's 1965 paper [3], by Bas Janssens.

1.1 Introduction

Let M be a smooth connected manifold and let Vec be the sheaf of smooth vector fields on M. Let $\mathcal{L} \subseteq \mathrm{Vec}$ be a sheaf of Lie algebras of vector fields, that is, a subsheaf of Vec that is closed under the Lie bracket. For $x \in M$, let $\mathcal{L}_x$ be the Lie algebra of germs of sections of $\mathcal{L}$ around x. It is filtered by the subalgebras $\mathcal{L}_{x,k}$ of germs that vanish to order k (where we set $\mathcal{L}_{x,k} = \mathcal{L}_x$ for $k < 0$).

Definition 1.1. We define the *formal Lie algebra* of $\mathcal{L}$ at x by

$$L_x := \varprojlim_k L_x^k,$$

where $L_x^k := \mathcal{L}_x/\mathcal{L}_{x,k}$ is the space of k-jets of sections of $\mathcal{L}$ at x.

It is filtered by the subalgebras $L_{x,k} := \varprojlim_n \mathcal{L}_{x,k}/\mathcal{L}_{x,n}$ of formal jets that vanish to order k. If we endow L_x with the inverse limit topology induced by the norm topology of the (finite dimensional) spaces L_x^k, then L_x becomes a Fréchet Lie algebra. For every open neighbourhood U of x, the Lie algebra $\mathcal{L}(U)$ is a (not necessarily closed) locally convex subalgebra of $\mathrm{Vec}(U)$, and the evaluation $\mathrm{ev}_x \colon \mathcal{L}(U) \to L_x$ is a morphism of locally convex Lie algebras.

Definition 1.2. We define $\mathfrak{g}_x$ to be the associated graded Lie algebra of L_x.

More explicitly, if we define $\mathfrak{g}_x^k$ to be the kernel of the map $L_x^{k+1} \to L_x^k$, that is, the space of $k+1$-symbols of sections of $\mathcal{L}$ at x, then we have $\mathfrak{g}^k = L_{x,k}/L_{x,k+1}$ (in particular, $\mathfrak{g}_x^k = \{0\}$ for $k < -1$), so

$$\mathfrak{g}_x = \prod_{k=-1}^{\infty} \mathfrak{g}^k.$$

With respect to the Lie algebra and topology inherited from L_x, the associated graded Lie algebra $\mathfrak{g}_x$ is a graded Fréchet Lie algebra.

An important example is obtained by taking $\mathcal{L} = \mathrm{Vec}$. Then L_x is $J_x^\infty := \varprojlim_k J_x^k(TM)$, the inverse limit of the k-jets at x of sections of the tangent bundle. Note that $J^\infty \to M$ is a locally trivial bundle of Fréchet Lie algebras.

We denote the associated graded Lie algebra $\mathfrak{g}_x$ by Sb_x, the inverse limit over the spaces $\mathrm{Sb}^k_x = \mathrm{Ker}(J^{k+1}_x(TM) \to J^k_x(TM))$, equal to $\mathrm{Vec}_{x,k}/\mathrm{Vec}_{x,k+1}$, of symbols of order $k+1$. Note that Sb^k_x can be canonically identified with $S^{k+1}(T^*_xM) \otimes T_xM$, and that this identification is a bundle isomorphism. We thus have

$$\mathrm{Sb} = \prod_{k=-1}^{\infty} S^{k+1}(T^*M) \otimes TM$$

as a locally trivial bundle of graded Fréchet Lie algebras over M.

1.2 Some relevant algebra

In this section, we will see that the fact that L_x is subalgebra of Sb_x imposes severe restrictions on the algebraic structure of L_x.

The coadjoint representation of Sb_x on its continuous dual

$$\mathrm{Sb}^*_x = \bigoplus_{k=-1}^{\infty} S^{k+1}(T_xM) \otimes T^*_xM$$

yields an action of the universal enveloping algebra $\mathcal{U}(\mathrm{Sb}_x)$ on Sb^*_x. The inclusion of the abelian Lie algebra $T_xM = \mathrm{Sb}^{-1}_x$ into Sb_x yields an inclusion of $\mathcal{U}(\mathrm{Sb}^{-1}_x) = S(T_xM)$ into $\mathcal{U}(\mathrm{Sb}_x)$, hence an action of $S(T_xM)$ on Sb^*_x.

Proposition 1.3. *The coadjoint action of $S(T_xM) \subseteq \mathcal{U}(\mathrm{Sb}_x)$ on the continuous dual $\mathrm{Sb}^*_x = S(T_xM) \otimes T^*_xM$ is induced by the symmetric tensor product $\vee \colon S(T_xM) \times S(T_xM) \to S(T_xM)$.*

Proof. For $v \in T_xM$, the adjoint action $\mathrm{ad}_v \colon \mathrm{Sb}^k_x \to \mathrm{Sb}^{k-1}_x$ is given on $\gamma \otimes w \in S^{k+1}(T^*M) \otimes T_xM \simeq \mathrm{Sb}^k_x$ by $\mathrm{ad}_v(\gamma \otimes w) = (i_v\gamma) \otimes w$. The dual of the annihilation operator i_v on $\overline{S}(T^*_xM) := \prod_{j=0}^{\infty} S^j(T^*_xM)$ is the creation operator $u \mapsto v \vee u$ on $S(T_xM)$. Since $\mathcal{U}(\mathrm{Sb}^{-1}_x)$ is generated by T_xM, the proposition follows. □

Corollary 1.4. *The annihilator $\mathrm{ann}(\mathfrak{g}_x)$ of $\mathfrak{g}_x$ in Sb^*_x is a $S(\mathfrak{g}^{-1}_x)$-module.*

Proof. For $v \in \mathfrak{g}^{-1}_x \subseteq T_xM$, we have $\mathrm{ad}_v(\mathfrak{g}) \subseteq \mathfrak{g}$. The action of ad^*_v on Sb^*_x therefore preserves the subspace $\mathrm{ann}(\mathfrak{g}_x)$. □

We call $\mathcal{L}$ *transitive* at $x \in M$ if $\mathfrak{g}^{-1}_x = T_xM$. Combining Corollary 1.4 with the Hilbert Basis Theorem and the Artin-Rees Lemma, we obtain the following theorem.

Theorem 1.5. *If $\mathcal{L}$ is transitive at $x \in M$, then there exists a $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$, the space $\mathfrak{g}^k_x \subseteq S^{k+1}(T^*_xM) \otimes T_xM$ is determined by $\mathfrak{g}^{k-1}_x \subseteq S^k(T^*_xM) \otimes T_xM$ by*

$$\mathfrak{g}^k_x = \left(T^*_xM \otimes \mathfrak{g}^{k-1}_x\right) \cap \left(S^{k+1}(T^*_xM) \otimes T_xM\right) .$$

Proof. Since $\mathfrak{g}^{-1}_x = T_xM$, Corollary 1.4 implies that $\mathrm{ann}(\mathfrak{g}_x)$ is a $S(T_xM)$-submodule of the finitely generated $S(T_xM)$-module $\mathrm{Sb}^*_x = S(T_xM) \otimes T^*_xM$. By the Hilbert Basis Theorem, $\mathrm{ann}(\mathfrak{g}_x)$ is finitely generated itself, and the Artin-Rees Lemma then implies that the filtration of $\mathrm{ann}(\mathfrak{g}_x)$ is essentially $\mathfrak{m}$-adic for the maximal ideal $\mathfrak{m} = \bigoplus_{j=1}^{\infty} S(T_xM)$ of $S(T_xM)$. Since $S(T_xM)$ and $\mathfrak{g}_x$ are graded, there exists a $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$, we have

$$\mathrm{ann}(\mathfrak{g}^k) = T_xM \vee \mathrm{ann}(\mathfrak{g}_x^{k-1})\,. \tag{1}$$

This implies that for sufficiently large values of k, the space $\mathfrak{g}_x^{k-1}$ determines $\mathfrak{g}^k$. Dualising equation (1), we calculate

$$\begin{aligned} \mathfrak{g}_x^k &= \mathrm{ann}(T_xM \otimes \mathrm{ann}(\mathfrak{g}_x^{k-1})) \cap S^{k+1}(T_x^*M) \otimes T_xM \\ &= T_x^*M \otimes \mathfrak{g}_x^{k-1} \cap S^{k+1}(T_x^*M) \otimes T_xM\,, \end{aligned}$$

where $\mathrm{ann}(T_xM \otimes \mathrm{ann}(\mathfrak{g}_x^{k-1}))$ is the annihilator of $T_xM \otimes \mathrm{ann}(\mathfrak{g}_x^{k-1})$ inside the tensor algebra $T^{k+1}(T_x^*M) \otimes T_xM$. □

In order to formulate this more succinctly, we introduce the notion of a prolongation.

Definition 1.6. The *prolongation* of a vector space $U \subseteq \mathrm{Hom}(V, W)$ is the subspace $U^{(1)} \subseteq \mathrm{Hom}(V, U)$ defined by

$$U^{(1)} = \{T \in \mathrm{Hom}(V, U)\,;\, T_u(v) = T_v(u)\,\forall\, u, v \in V\}\,.$$

If we identify $\mathfrak{g}^{k-1} \subseteq S^k(T_x^*M) \otimes T_xM$ as a subspace of $\mathrm{Hom}(T_xM, S^{k-1}(T_x^*M) \otimes T_xM)$, then Theorem 1.5 can be reformulated as follows: if $\mathcal{L}$ is transitive at $x \in M$, then for sufficiently large values of k, we have $\mathfrak{g}^k = (\mathfrak{g}^{k-1})^{(1)}$. In this language, we obtain the following corollary from the proof of Theorem 1.5.

Corollary 1.7. *If $\mathcal{L}$ is transitive at $x \in M$, then $\mathfrak{g}_x^k \subseteq (\mathfrak{g}_x^{k-1})^{(1)}$ for all $k \in \mathbb{N}$.*

Proof. By Corollary 1.4, $\mathrm{ann}(\mathfrak{g}_x)$ is a $S(T_xM)$-module, implying $\mathrm{ann}(\mathfrak{g}_x^k) \supseteq T_xM \vee \mathfrak{g}_x^{k-1}$ for all $k \in \mathbb{N}$. Repeating the dualisation at the end of the proof of Theorem 1.5, we obtain $\mathfrak{g}_x^k \subseteq \big(T_x^*M \otimes \mathfrak{g}_x^{k-1}\big) \cap \big(S^{k+1}(T_x^*M) \otimes T_xM\big)$. □

We can drop the requirement that $\mathcal{L}$ be transitive at $x \in M$ if we assume the following property:

Definition 1.8. The graded Lie algebra $\mathfrak{g}_x$ is a *tower of tableaux starting at* k_0 if $\mathfrak{g}_x^{r+1} \subseteq (\mathfrak{g}_x^r)^{(1)}$ for all $r \geq k_0$.

Note that every transitive $\mathfrak{g}_x$ is a tower of tableaux starting at 0 by the previous corollary. If $\mathcal{L}$ is defined by a regular PDE of order k_0, then it is a tower of tableaux starting at k_0.

Corollary 1.9. *Let $\mathcal{L}$ be such that $\mathfrak{g}_x$ is a tower of tableaux starting at k_0. Then there exists a $k \geq k_0$ such that $\mathfrak{g}_x^{r+1} = (\mathfrak{g}_x^r)^{(1)}$ for all $r \geq k$.*

Proof. The assumption that $\mathfrak{g}_x^{r+1}$ be contained in $(\mathfrak{g}_x^r)^{(1)}$ for all $r \geq k_0$ is equivalent to $\bigoplus_{r=k_0}^{\infty} \mathrm{ann}(\mathfrak{g}_x^r)$ being an $S(T_xM)$-submodule of Sb_x^*. The proof of Theorem 1.5, mutatis mutandis, then yields the required result. □

Definition 1.10. We will say that $\mathcal{L}$ is of order k if $\mathfrak{g}_x^{r+1} = (\mathfrak{g}_x^r)^{(1)}$ for all $r \geq k$.

1.3 Symmetries of a singular distribution

The k^{th} *order frame bundle* $F^k(M)$ is the manifold of k-jets at zero of local diffeomorphisms ψ from $U \subseteq \mathbb{R}^n$ to $\psi(U) \subseteq M$. Equipped with the projection $\pi\colon F^k(M) \to M$ defined by $\pi(j^k_0\psi) := \psi(0)$, it becomes a principal fibre bundle with structure group $\mathrm{Gl}^k(n)$, the group of k-jets of diffeomorphisms of $\mathbb{R}^n$ that fix 0.

It carries an action by bundle automorphisms of the diffeomorphism group $\mathrm{Diff}(M)$, defined by $\phi\colon j^k_0\psi \mapsto j^k_0(\phi \circ \psi)$. This yields a Lie algebra homomorphism $F^k\colon \mathrm{Vec}(M) \to \mathrm{Vec}(F^k(M))$. Any sheaf $\mathcal{L}$ of Lie algebras of vector fields on M therefore gives rise to a sheaf $F^k\mathcal{L}$ of Lie algebras of vector fields on $F^k(M)$. This in turn gives rise to the (singular) distribution $\Delta^k \subseteq TF^kM$ of values of $F^k\mathcal{L}$.

Conversely, given a (singular) distribution $\Delta \subseteq TJ^k(M,M)$, we define the sheaf $\mathcal{L}_\Delta$ of infinitesimal symmetries of Δ by

$$\mathcal{L}_\Delta(U) := \{v \in \mathrm{Vec}(U)\,;\, \mathrm{Im}(F^k(v)) \subseteq \Delta\}\,.$$

Clearly, $\mathcal{L}$ is always a subsheaf of the sheaf of infinitesimal symmetries of the distribution Δ^k obtained from it, $\mathcal{L} \subseteq \mathcal{L}_{\Delta^k}$.

Let $J^k(M,M) \rightrightarrows M$ be the groupoid of k-jets of local diffeomorphisms of M. Since this is a transitive groupoid with a canonical left action on $F^k(M)$ (defined by $j^k_x\phi\colon j^k_x\psi \mapsto j^k_x(\phi\circ\psi)$), a choice f_x of k-frame yields an identification of $F^k(M) \to M$ with the source fibre of $J^k(M,M) \rightrightarrows M$ over x, and of $J^k(M,M) \rightrightarrows M$ with the gauge groupoid $F^k(M) \times F^k(M)/\mathrm{Gl}^k(n) \rightrightarrows M$.

The action of $\mathrm{Diff}(M)$ on $F^k(M)$ factors through the canonical splitting homomorphism

$$\Sigma\colon \mathrm{Diff}(M) \to \mathrm{Bis}(J^k(M,M))\;;\; \Sigma(\phi)_x := j^k_x\phi$$

into the group $\mathrm{Bis}(J^k(M,M))$ of bisections. We identify $J^k(TM) \to M$ with the Lie algebroid of the groupoid $J^k(M,M)$ (that is, the pull back by the identity $e\colon M \to J^k(M,M)$ of the kernel $T^sJ^k(M,M)$ of the differential $s_*\colon TJ^k(M,M) \to TM$ of the source map s). Then the Lie algebra homomorphism $\mathrm{Vec}(M) \to \Gamma(J^k(TM))$ induced by the splitting homomorphism Σ is precisely the map $v \mapsto j^k(v)$. This shows that the distribution Δ^k on $F^k(M)$ is the image of $L^k \subseteq J^k(TM)$ under the Lie algebroid action $\pi^*J^k_x \to TF^k(M)$. In particular, $F^k(v)_{f_x}$ is in $\Delta^k_{f_x}$ if and only if j^kv is in L^k_x.

We will use this in following lemma, which says that under mild conditions on $\mathcal{L}$, the sheaves $\mathcal{L}$ and $\mathcal{L}_{\Delta^k}$ are in fact identical.

Lemma 1.11. *Let $\mathcal{L}$ be a sheaf of Lie algebras of vector fields such that $\mathfrak{g}_x$ is a tower of tableaux starting at k_0 for all $x \in M$. Suppose that the order of L_x is bounded by k on M. Suppose also that $\mathfrak{g}_{\Delta^k}$ is a tower of tableaux starting at k. Then we have $L = L_{\Delta^k}$.*

If, moreover, $\mathcal{L}$ is determined by L in the sense that every $v \in \mathrm{Vec}(U)$ with $j^\infty_x(v) \in L_x$ for all $x \in U$ belongs to $\mathcal{L}(U)$, then we have $\mathcal{L} = \mathcal{L}_{\Delta^k}$.

Proof. Since $\mathcal{L}$ is contained in $\mathcal{L}_{\Delta^k}$, we have $L \subseteq L_{\Delta^k}$. Because v is in $\mathcal{L}_{\Delta^k}(U)$ if and only if $F^k(v)_{f_x}$ is in $\Delta^k_{f_x}$ for all $f_x \in F^k(M)$, which in turn is the case if and only if $j^k_x(v) \in L^k_x$ for all $x \in U$, we have $L^k = L^k_{\Delta^k}$. In particular, $\mathfrak{g}^r = \mathfrak{g}^r_{\Delta^k}$

for all $r \leq k$. Now $\mathfrak{g}^{k+r}_{\Delta^k} \subseteq (\mathfrak{g}^k_{\Delta^k})^{(r)}$ for all $r > 0$ by assumption. We thus find

$$\mathfrak{g}^{k+r}_{\Delta^k} \subseteq (\mathfrak{g}^k_{\Delta^k})^{(r)} = (\mathfrak{g}^k)^{(r)} = \mathfrak{g}^{k+r}\,,$$

the last equality following from the fact that $\mathfrak{g}$ is of order at most k. Since the opposite inclusion is clear, we have $\mathfrak{g}^r_{\Delta^k} = \mathfrak{g}^r$ for all r. From this and from $L \subseteq L_{\Delta^k}$, it follows that $L = L_{\Delta^k}$.

It remains to prove the last statement. We have already seen that $\mathcal{L} \subseteq \mathcal{L}_{\Delta^k}$ and $L = L_{\Delta^k}$. Now every $v \in \mathcal{L}_{\Delta^k}(U)$ satisfies $j^\infty_x(v) \in (L_{\Delta^k})_x = L_x$ for all $x \in M$, hence lies in $\mathcal{L}(U)$. Thus $\mathcal{L}_{\Delta^k} \subseteq \mathcal{L}$ and the sheaves are equal. □

The problem of realising $\mathcal{L}$ as the sheaf of infinitesimal symmetries of a finite dimensional geometric object is thus essentially equivalent to the problem of integrating the singular foliation Δ^k.

By the Stefan-Sussman theorem [4, Corollary 1], the singular distribution Δ^k is integrable (through every point in $J^k(M,M)$ passes an intgral manifold of Δ^k) if and only if the pushforward $\exp(tX)_*\colon L^k_x \to L^k_y$ of the local flow $\exp(tX)$ along $X \in \mathcal{L}(U)$ is an isomorphism for all $x \in \mathrm{Dom}(\exp(tX))$ and $y = \exp(tX)(x)$.

The assumption that $\mathfrak{g}_{\Delta^k}$ be a tower of tableaux is in this context a plausibe assumption; if Δ^k is sufficiently regular, it follows from the fact that $\mathcal{L}^k_\Delta$ is defined by a PDE of order k.

Example 1.12. Let $\mathcal{L}$ be the sheaf of vector fields on $\mathbb{R}^n$ that vanish at 0. Then one readily checks that $\mathcal{L}$ satisfies the conditions of Lemma 1.11 for $k = 0$. The distribution Δ^0_x on $F^0(M) = M$ is given by $T_x\mathbb{R}^n$ for $x \neq 0$ and by $\{0\}$ for $x = 0$. It is clearly integrable and yields the singular foliation of $\mathbb{R}^n$ into 0 and $\mathbb{R}^n - 0$. Lemma 1.11 thus constructs a geometric object (the division of $\mathbb{R}^n$ into $\{0\}$ and $\mathbb{R}^n - \{0\}$) from the sheaf $\mathcal{L}$, and yields the somewhat tautologous statement that every vector field with $j^0v|_0 = 0$ belongs to $\mathcal{L}$.

Example 1.13. Let $\mathcal{L}$ be the sheaf of vector fields on $\mathbb{R}^n$ that vanish on the x^1-axis ℓ. Then $\mathfrak{g}_x = \mathrm{Sb}_x$ for $x \notin \ell$ and $\mathfrak{g}_x = \sum_{i=2}^n x_i \cdot \mathrm{Sb}_x$ for $x \in \ell$. Hence $(\mathfrak{g}^k)^{(1)}$ is strictly smaller than $\mathfrak{g}^{k+1}$ for all k, and Lemma 1.11 does not apply. Nonetheless, the distribution Δ^0 is well defined and integrable, and the foliation of $\mathbb{R}^n$ into $\mathbb{R}^n - \ell$ and the points $\{p\} \in \ell$ determines the sheaf.

1.4 Lie Algebra Sheaves (LAS) of order k

We streamline the process of applying Lemma 1.11 by imposing conditions on $\mathcal{L}$ that insure regularity of Δ^k. We call a sheaf of Lie algebras *regular of order* k if $L^k \to M$ is a smooth vector bundle.

Definition 1.14. We call a sheaf $\mathcal{L}$ of Lie algebras of vector fields a *Lie Algebra Sheaf* (LAS) of order k if it is regular of order k, if $\mathfrak{g}_x$ is a tower of tableaux starting at $k_0 \leq k$ and the order of L_x is bounded by k on M, and if $\mathcal{L}$ is determined by L in the sense that $j^\infty_x v \in L_x$ for all $x \in U$ implies $v \in \mathcal{L}(U)$.

Remark 1.15. Every sheaf of Lie algebras of vector fields defined by a regular, linear PDE of order k_0 is a LAS of order k for some $k \geq k_0$.

Remark 1.16. Part of the definition of a LAS in the sense of Singer-Sternberg [3, Def. 1.8] is regularity of order 0. Our notion of a LAS of order k is therefore less restrictive.

The following theorem, which is essentially *Cartan's First Fundamental Theorem*, is a reformulation of Lemma 1.11 for LAS of order k.

Theorem 1.17 (Cartan I for LAS of order k)**.** *For every LAS of order k, there exists a Lie groupoid $\mathcal{G}_L^k \rightrightarrows M$ with a locally free action*

$$a\colon \mathcal{G}_L^k \ {}_s\!\times_\pi F^k(M) \to F^k(M)$$

such that $\mathcal{L}$ is the sheaf of symmetries of this action, in the sense that $\mathcal{L}(U)$ is the Lie algebra of all $v \in \mathrm{Vec}(M)$ such that the vector field $F^k(v)$ on $F^k(M)$ is parallel to the $\mathcal{G}_L^k$-orbits.

In other words: every LAS of order k is the sheaf of infinitesimal symmetries of a locally free groupoid action on the k^{th} order frame bundle.

Proof. The fact that $L^k \to M$ is a smooth vector bundle implies that Δ^k is a regular foliation. In particular, $\mathcal{L}_{\Delta^k}$ is defined by a PDE of order k, so that $\mathfrak{g}_{\Delta^k}$ is a tower of tableaux starting at k. It follows from Lemma 1.11 that $\mathcal{L} = \mathcal{L}_{\Delta^k}$.

Since L^k is smooth and $\mathcal{L}$ is closed under the Lie bracket, $L^k \to M$ is a Lie subalgebroid of the integrable algebroid $J^k(TM)$. By Prop. 3.4 and 3.5 in [1], L^k then integrates to a Lie groupoid $\mathcal{G}_L^k \rightrightarrows M$ with an immersive morphism $\iota\colon \mathcal{G}_L^k \to J^k(M,M)$ of Lie groupoids. (In general, this immersion will be neither injective nor closed [2].) The free action of $J^k(M,M)$ on $F^k(M)$ then yields a locally free action of $\mathcal{G}_L^k$ on $F^k(M)$, and v is in $\mathcal{L}_{\Delta^k}(U)$ if and only if $F^k(v)$ is tangent to the orbits. □

The following is a simple example of a LAS of order 1 which is not a LAS in the sense of Singer-Sternberg. This shows that Theorem 1.17 applies to a trictly wider class of sheaves than the ones in [3].

Example 1.18. Let $\mathcal{L}$ be the sheaf of Lie algebras of vector fields on $M = \mathbb{R}^2$ defined by letting $\mathcal{L}(U) := \mathbb{R}\cdot v|_U$ for $v := x_1\partial_{x_2} - x_2\partial_{x_1}$. If we identify J_x^∞ with $\prod_{k=-1}^{\infty} S^{k+1}(\mathbb{R}^{2*}) \otimes \mathbb{R}^2$, we obtain

$$L_{(u_1,u_2)} = \mathbb{R}\cdot \left((u_1\partial_{x_2} - u_2\partial_{x_1}) \oplus (dx_1 \otimes \partial_{x_2} - dx_2 \otimes \partial_{x_1})\right).$$

Note that $L^1 \to M$ is a smooth bundle, as are all the L^k with $k \geq 1$ and the bundle of Fréchet spaces $L \to M$. The sheaf $\mathcal{L}$ is a LAS of order 1 because, moreover, $\mathfrak{g}_x^k = 0$ for $k \geq 1$.

Note, however, that none of the maps $L^0 \to M$, $\mathfrak{g}^{-1} \to M$ and $\mathfrak{g}^0 \to M$ have constant rank, so $\mathcal{L}$ is not regular of order 0. The groupoid $\mathcal{G}_L^1$ integrating the Lie algebroid $L^0 \to M$ is easily seen to be the action groupoid $\mathbb{R}^2 \times \mathrm{SO}(2) \rightrightarrows \mathbb{R}^2$ with the obvious action on the frame bundle $F^1(\mathbb{R}^2)$.

This is the general situation for sheaves of Lie algebras that come from group actions with finite order fixed points.

Corollary 1.19 (Symmetries of a group action)**.** *Let G be a connected Lie group, $G \curvearrowright M$ a Lie group action, and $\xi\colon \mathfrak{g} \to \mathrm{Vec}(M)$ the associated Lie algebra morphism. Suppose that for every nonzero $X \in \mathfrak{g}$, all fixed points of ξ_X are of order $\leq k_0$. Then $\mathcal{L}(U) := \{\xi_X|_U\,;\, X \in \mathfrak{g}\}$ is a LAS of order $k_0 + 1$. The corresponding groupoid $\mathcal{G}_L^k$ is the action groupoid $G \times M \rightrightarrows M$ with the obvious action on $F^k(M)$.*

Proof. Since the fixed points are of order k_0, the action of $G \times M$ on $F^k(M)$ is locally free for $k \geq k_0 + 1$. Because the vector fields $F^k(\xi_X)$ on $F^k(M)$ are nonvanishing, $\mathcal{L}$ is a sheaf and L^k is a smooth subbundle of $J^k(TM)$. Since $\mathfrak{g}_x^r$ is the kernel of $L^{r+1} \to L^r$, it is zero for $r \geq k$. It follows that $\mathcal{L}$ is a LAS of order k. The groupoid integrating the Lie algeboid L^k is the action groupoid with the canonical action on $F^k(M)$, so Theorem 1.17 implies that $\mathcal{L}$ is the sheaf of vector fields v such that for each k-frame f_x, there exists an $X \in \mathfrak{g}$ such that $F^k(v)_{f_x} = F^k(\xi_X)_{f_x}$. □

1.5 Transitive sheaves

The situation becomes especially transparent if the sheaves of Lie algebras are transitive.

Definition 1.20. Let $\mathcal{L}$ be a sheaf of Lie algebras of vector fields on a connected manifold M. Then $\mathcal{L}$ is called a *transitive Lie Algebra Sheaf* if $L_x^0 = T_xM$ for all $x \in M$, if the pushforward $\exp(tX)_* \colon \mathcal{L}_x \to \mathcal{L}_y$ yields an isomorphism $L_x \to L_y$ for all $x, y \in M$ with $y = \exp(tX)(x)$ and if $\mathcal{L}$ is determined by L in the sense that $j_x^\infty(v) \in L_x$ for all $x \in U$ implies $v \in \mathcal{L}(U)$.

This coincides with the notion of a transitive LAS in the sense of Singer-Sternberg, cf. [3], def. 1.3, 1.4 and 1.8. We set out to prove that every transitive LAS is a LAS of finite order in the sense defined before.

Proposition 1.21. *For every transitive LAS, the subset $L \subseteq J^\infty$ is a smooth locally trivial bundle of Fréchet Lie algebras over M. The same holds for all bundles $L^k \to M$.*

Proof. Since $\mathcal{L}$ is transitive, every $x_0 \in M$ possesses a neighbourhood U with a local frame $X_1, \ldots, X_d$ of sections of $\mathcal{L}(U)$. For $\vec{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$ sufficiently close to zero, $\phi_{\vec{x}} = \exp(x_1X_1) \circ \ldots \circ \exp(x_nX_n)$ is well defined. By shrinking U if necessary, we obtain a chart $\kappa^{-1} \colon \mathbb{R}^d \supseteq V \to U \subseteq M$ by $\kappa^{-1}(\vec{x}) = \phi_{\vec{x}}(x_0)$. The map $V \times L_{x_0} \to L$ defined by $(\vec{x}, Y) \mapsto \phi_{\vec{x}*}Y$ is a local trivialisation of L over U and every two such trivialisations differ by a smooth isomorphism of the trivial bundles $V \cap \kappa\kappa'^{-1}(V') \times L_{x_0} \to V' \cap \kappa'\kappa^{-1}(V) \times L_{x_0'}$. Since these isomorphisms preserve the filtration, not only $L \to M$ but also all the $L^k \to M$ are smooth. □

By Theorem 1.5, transitivity of L_x, i.e. the requirement $\mathfrak{g}_x^{-1} = T_xM$ for all $x \in M$, implies that L_x has finite order. Because all L_x are isomorphic, the order is locally constant, hence finite. We arrive at the following proposition.

Proposition 1.22. *Every transitive LAS is a LAS of finite order that is regular of order* 0*. For all $k \in \mathbb{N}$, the bundle $L^k \to M$ is a transitive Lie algebroid.*

We can now apply Theorem 1.17 to obtain a *transitive* groupoid $\mathcal{G}_L^k$, the source fibre at of which is a principal fibre bundle over M.

Theorem 1.23 (Cartan I for Transitive LAS)**.** *If $\mathcal{L}$ is a transitive LAS, then there exists an immersed principal subbundle $P_L \subseteq F^k(M)$ whose structure group has Lie algebra $L_{x,0}/L_{x,k+1}$, such that $\mathcal{L}$ is the sheaf of infinitesimal symmetries of P_L, in the sense that $v \in \mathrm{Vec}(U)$ is in $\mathcal{L}(U)$ if and only if $F^k(v)|_p$ is in T_pP_L for all $p \in P_L$.*

In other words: every transitive LAS is the sheaf of symmetries of an immersed subbundle of the k^{th} order frame bundle.

Proof. Since $\mathcal{L}^k$ is transitive, so is $\mathcal{G}_L^k \rightrightarrows M$, and its source fibre at $x_0 \in M$ is the principal fibre bundle $\mathcal{G}^k_{L,*,x_0} \to M$. The Lie algebra of its structure group $\mathcal{G}^k_{L,x_0,x_0}$ is the kernel of the anchor $L^k_{x_0} \to L^0_{x_0}$. Since $\iota\colon \mathcal{G}^k_L \to J^k(M,M)$ is an immersion, the kernel K of the map $\iota\colon \mathcal{G}_{L,x_0,x_0} \to \mathcal{G}^k_{x_0,x_0}$ is a *closed* discrete normal subgroup of $\mathcal{G}_{L,x_0,x_0}$. Now $P := \mathcal{G}^k_{L,*,x_0}/K$ is a principal fibre bundle with structure group $\mathcal{G}^k_{L,x_0,x_0}/K$. If we identify the source fibre at x_0 of $J^k(M,M)$ with $F^k(M)$, then the equivariant immersion $P \to F^k(M)$ is injective, and the distribution Δ^k on $F^k(M)$ consists of the translates of $TP \subseteq TF^k(M)$ by the structure group $\mathrm{Gl}^k(n)$. □

Let us consider the images P^j_L of P_L under the projection maps $\pi^k_j\colon F^k(M) \to F^j(M)$. For each $j = 0, \ldots, k$, we have an inclusion $\iota_j\colon P^j_L \hookrightarrow F^j(M)$. The immersed subbundle P^1_L of $F^1(M)$ is a subbundle of the 'ordinary' frame bundle, hence a G-structure on M. Its structure Lie algebra is $\mathfrak{g}^0_{x_0}$.

We can consider $P^2_L \subseteq F^2(M)$ as a principal fibre bundle over P^1_L with structure Lie algebra $\mathfrak{g}^1_{x_0}$, but also as a subbundle of the prolongation $(P^1_L)^{(1)} \to P^1_L$ with structure Lie algebra $\mathfrak{g}^1 \subseteq \mathfrak{g}^{(1)}$. Continuing in this way, we obtain a tower of principal fibre bundles

$$P^k_L \to \cdots \to P^1_L \to M$$

such that $P^k_L = P_L$ and $P^{j+1}_L \to P^j_L$ has structure Lie algebra $\mathfrak{g}^j_{x_0}$. This can be regarded as a higher order G-structure.

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