A double complex for the Poisson Lie algebra

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Abstract

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K}$. We describe a double complex $\{\mathcal{C}^{\bullet,\bullet}\}$ of $\mathbb{K}$-vector spaces that gives rise to a spectral sequence relating the cohomology rings $H^{\bullet}_{\mathrm{LA}}(\mathfrak{g}, S^m(\mathfrak{g}^*))$ for different m. We apply this to the Poisson Lie algebra.

1 The Weyl complex

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K}$. We describe a double complex $\{\mathcal{C}^{\bullet,\bullet}\}$ of $\mathbb{K}$-vector spaces that gives rise to a spectral sequence relating the cohomology rings $H^{\bullet}_{\mathrm{LA}}(\mathfrak{g}, S^m(\mathfrak{g}^*))$ for different m.

For $n \geq m$, we set $A_{n,m} := \Lambda^{n-m}\mathfrak{g} \otimes S^m\mathfrak{g}$ and define $\mathcal{C}^{n,m} := A^*_{n,m} = C^{n-m}_{\mathrm{LA}}(\mathfrak{g}, S^m\mathfrak{g}^*)$. Note that this is nonzero only for $m \geq 0$, $n \geq m$, so that the complex is concentrated on the second octant. Note also that $\bigoplus_{\mathbb{Z}^2} \mathcal{A}_{n,m}$ is isomorphic to the tensor product $\Lambda\mathfrak{g} \otimes S\mathfrak{g}$ of the fermionic and bosonic Fock space over $\mathfrak{g}$, where n is the total number of particles and m is the boson number.

The vertical differential $\delta_{LA} \colon \mathcal{C}^{n,m} \to \mathcal{C}^{n+1,m}$ is the differential of Lie algebra cohomology. It is the dual of the map $\delta^m_* \colon A^{n+1,m} \to A^{n,m}$ given by

$$\begin{aligned}\delta^m_*(x_0 \wedge \ldots \wedge x_{n-m}) \otimes Y &= (\delta^0_* x_0 \wedge \ldots \wedge x_{n-m}) \otimes Y \\ &\quad + \sum_{i=0}^{n-m} (-1)^i (x_0 \wedge \ldots \wedge \hat{x}_i \wedge \ldots \wedge x_{n-m})(-\mathrm{ad}_{x_i}(Y)),\end{aligned}$$

where $\mathrm{ad}_x(y_1 \vee \ldots \vee y_m) := \sum_{j=1}^m y_1 \vee \ldots \vee [x, y_j] \vee \ldots \vee y_m$ is the representation of $\mathfrak{g}$ on $S^m\mathfrak{g}$ and

$$\delta^0_* x_0 \wedge \ldots \wedge x_n := \sum_{0 \leq i < j \leq n} (-1)^{i+j} [x_i, x_j] \wedge \ldots \wedge \hat{x}_i \wedge \ldots \wedge \hat{x}_j \wedge \ldots \wedge x_n$$

is the Lie algebra differential with trivial coefficients.

The horizontal differential $\alpha \colon C^{n,m} \to C^{n,m+1}$ is defined as the dual of the antisymmetrisation map $\alpha_* \colon A^{n,m+1} \to A^{n,m}$ defined by

$$(x_1 \wedge \ldots \wedge x_{n-m-1}) \otimes (y_0 \vee \ldots \vee y_m) \mapsto \sum_{i=0}^{m} (x_1 \wedge \ldots \wedge x_{n-m-1} \wedge y_i) \otimes (y_0 \vee \ldots \vee \hat{y}_i \vee \ldots \vee y_m).$$

One checks that $\alpha^2_* = 0$, for instance by noting that the (i,j) and (j,i) terms in

$$\alpha^2_* X \otimes (y_0 \vee \ldots \vee y_m) = \sum_{i \neq j} (X \wedge y_j \wedge y_i) \otimes (y_1 \vee \ldots \vee \hat{y}_i \vee \ldots \vee \hat{y}_j \vee \ldots y_m)$$

cancel.

Lemma 1.1. *The horizontal and vertical differentials α and δ_{LA} on $\mathcal{C}^{\bullet,\bullet}$ constitute a double complex, $\delta_{LA} \circ \alpha - \alpha \circ \delta_{LA} = 0$.*

Proof. We show by explicit calculation that $\delta_* \circ \alpha_* = \alpha_* \circ \delta_*$, from which the dual statement in the theorem then follows. For notational convenience, we write $X = x_0 \wedge \ldots \wedge x_{n-m}$, and $X^i = x_0 \wedge \ldots \wedge \hat{x}_i \wedge \ldots x_{n-m}$ for its ith face, X^{ij} where x_i and x_j are omitted, etc. Similarly, we write $Y = y_0 \vee \ldots \vee y_m$ and Y^s for the above with y_s omitted. Now

$$\alpha_* X \otimes Y = \sum_{s=0}^{m} X \wedge y_s \otimes Y^s$$

and

$$\begin{aligned}\delta_*\alpha_* X \otimes Y \quad = \quad & \sum_{s=0}^{m} \sum_{0 \leq i < j \leq n-m} (-1)^{i+j} [x_i, x_j] \wedge X^{ij} \wedge y_s \otimes Y^s \\ & + \sum_{s=0}^{m} \sum_{i=0}^{n-m} (-1)^{i+n-m+1} [x_i, y_s] \wedge X^i \otimes Y^s \\ & + \sum_{s=0}^{m} \sum_{i=0}^{n-m} (-1)^i X^i \wedge y_s \otimes (-\mathrm{ad}_{x_i} Y^s) \\ & + (-1)^{n-m+1} X \otimes \sum_{s=0}^{m} (-\mathrm{ad}_{y_s} Y^s)\,.\end{aligned}$$

The last term is proportional to $\sum_{s \neq t} [y_s, y_t] \vee Y^{st} = 0$, hence vanishes. In the first term, we recognise $\alpha_*(\delta^0_* X) \otimes Y$, and the second and third term combine to form $\alpha_*(\sum_{i=0}^{n+m} (-1)^i X^i \otimes (-\mathrm{ad}_{x_i} Y))$. This shows that $\delta_* \circ \alpha_* = \alpha_* \circ \delta_* 0$. □

Lemma 1.2. *For $\mathbb{K}$ equal to $\mathbb{R}$ or $\mathbb{C}$, the differential α is exact, so that the horizontal cohomology of the complex $\mathcal{C}^{\bullet,\bullet}$ vanishes.*

Proof. Every element $Z \in \Lambda^{n-m}\mathfrak{g} \otimes S^m\mathfrak{g}$ is a finite linear combination $Z = \sum_{k=1}^{N} X_k \otimes Y_k$ of pure tensors, hence contained in $A_{n,m}(V) := \Lambda^{n-m}V \otimes S^m V$, with $V \subseteq \mathfrak{g}$ the finite dimensional subspace spanned by the elements of $\mathfrak{g}$ that form the pure tensors $X_k \otimes Y_k$. It is therefore sufficient to prove the statement (which does not involve the Lie algebra structure) on finite dimensional vector spaces V over $\mathbb{K}$. We may assume w.l.o.g. that $\mathbb{K} = \mathbb{C}$, because the case $\mathbb{K} = \mathbb{R}$ follows by complexification.

Note that the map α_* is not identically zero if $A_{n,m} \neq \{0\}$, because with $X = x_1 \wedge \ldots \wedge x_{n-m-1} \neq 0$ and $Y = y^{\vee m+1}/(m+1)!$ for a vector y not contained in the span of the x_i, one has $\alpha_*(X \otimes Y) = (X \wedge y) \otimes y^{\vee m}/m! \neq 0$.

If V is of dimension d, then the map $\alpha_* \colon A_{n,m+1} \to A_{n,m}$ is an intertwiner of $\mathrm{Sl}_d(\mathbb{C})$-representations. Now $S^m V$ and $\Lambda^{n-m}V$ are irreducible, whereas for $0 < m < n$, the representation $\Lambda^{n-m}V \otimes S^m V$ decomposes as the direct sum of two irreducible representations, one with highest weight $(m+1)L_1 + L_2 + \ldots + L_{n-m}$ and one with highest weight $mL_1 + L_2 + \ldots + L_{n-m+1}$, where L_i is the weight of the i^{th} basis vector of the defining representation V. (See e.g. [FH91, §15, Prop. 15.25].)

Since $\alpha_*\colon A_{n,m+1} \to A_{n,m}$ is a nonzero intertwiner, it must be zero on the representation with highest weight $(m+2)L_1 + L_2 + \otimes + L_{n-m-1}$ and an isomorphism on the representation with weight $(m+1)L_1+L_2+\ldots+L_{n-m}$. But because $\alpha_*\colon A_{n,m} \to A_{n,m-1}$ is zero on the representation with highest weight $(m+1)L_1 + L_2 + \otimes + L_{n-m}$ and an isomorphism on the representation with weight $mL_1 + L_2 + \ldots + L_{n-m+1}$, the sequence $A_{n,m+1} \xrightarrow{\alpha_*} A_{n,m} \xrightarrow{\alpha_*} A_{n,m-1}$ is exact at $A_{n,m}$. □

A diagram chase through the lower part of the complex then yields:

Corollary 1.3. *For $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, there is an exact sequence*

$$0 \to H^2(\mathfrak{g}, \mathbb{K}) \to H^1(\mathfrak{g}, \mathfrak{g}^*) \to (S^2\mathfrak{g}^*)^{\mathfrak{g}} \to H^3(\mathfrak{g}, \mathbb{K}) \to H^2(\mathfrak{g}, \mathfrak{g}^*) \to H^1(\mathfrak{g}, S^2\mathfrak{g}^*)\,.$$

The sequence continues to $(S^3\mathfrak{g}^*)^{\mathfrak{g}}$ and beyond, but is no longer exact. For example, the cohomology of

$$H^2(\mathfrak{g}, \mathfrak{g}^*) \to H^1(\mathfrak{g}, S^2\mathfrak{g}^*) \to (S^3\mathfrak{g}^*)^{\mathfrak{g}} \tag{1}$$

in the middle term is not zero, but $H^4(\mathfrak{g}, \mathbb{K})/K$, with K the kernel of the map $\alpha\colon H^4(\mathfrak{g}, \mathbb{K}) \to H^3(\mathfrak{g}, \mathfrak{g}^*)$.

If we replace the differential α by $(-1)^{n-m-1}\alpha$ to make it anticommute with δ_{LA}, we obtain a spectral sequence from the double complex with which one can calculate the lack of exactness. It converges to zero because α_* is exact, so we obtain:

Corollary 1.4. *For $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, the spectral sequence associated to the complex $\mathcal{C}^{\bullet,\bullet}$ converges to zero, $H^p_\alpha(H^q_{\mathrm{LA}}\mathcal{C}^{\bullet,\bullet}) \Rightarrow_p 0$.*

The above statement on the sequence (1) can easily be derived from this.

2 The Poisson Algebra

Let $(R, \cdot, \{\cdot\,,\cdot\})$ be a commutative $\mathbb{K}$-algebra $(R, \cdot\,)$ with a Lie bracket satisfying $\{f, g \cdot h\} = g \cdot \{f, h\} + \{f, g\} \cdot h$. We assume the field $\mathbb{K}$ to be either $\mathbb{R}$ or $\mathbb{C}$, and all tensor products will be over $\mathbb{K}$ unless stated otherwise.

2.1 An intertwiner $\mu_*\colon S^m R \to S^{m-1}R$

Before, we have used, perhaps implicitly, the fact that both $\Lambda^{n-m}R$ and $S^m R$ are modules for the Lie algebra $(R, \{\cdot\,,\cdot\})$ under the actions

$$\mathrm{ad}_F\colon f_1 \wedge \cdots \wedge f_{m-n} \mapsto \sum_{i=1}^{m-n} f_1 \wedge \cdots \wedge \{F, f_i\} \wedge \cdots \wedge f_{m-n}$$

and

$$\mathrm{ad}_F\colon F_1 \vee \cdots \vee F_m \mapsto \sum_{i=1}^{m} F_1 \vee \cdots \vee \{F, F_i\} \vee \cdots \vee F_{m-n}\,.$$

respectively. Now, we note that both $\Lambda^{n-m}R$ and $S^m R$ are also modules for the algebra $(R, \cdot\,)$ under the respective actions

$$M_F\colon f_1 \wedge \cdots \wedge f_{m-n} \mapsto \sum_{i=1}^{m-n} f_1 \wedge \cdots \wedge F \cdot f_i \wedge \cdots \wedge f_{m-n}$$

and

$$M_F\colon F_1 \vee \cdots \vee F_m \mapsto \sum_{i=1}^{m} F_1 \vee \cdots \vee F \cdot F_i \vee \cdots \vee F_{m-n}\,.$$

The actions of $(R,\,\cdot\,)$ and $(R,\{\,\cdot\,,\,\cdot\,\})$ are intertwined by

$$\mathrm{ad}_F \circ M_G - M_G \circ \mathrm{ad}_F = M_{\{F,G\}}\,. \tag{2}$$

For $m \geq 1$, we define the map

$$\mu_*\colon S^m R \to S^{m-1}R$$

by

$$\mu_*\colon F_1 \vee \ldots \vee F_m \mapsto \sum_{i=1}^{m} M_{F_i}(F_1 \vee \ldots \vee \widehat{F}_i \vee \ldots \vee F_m)\,.$$

This equals

$$\mu_*\colon (F_1 \vee \ldots \vee F_m) = 2\sum_{i<j} F_iF_j \vee F_1 \vee \ldots \vee \widehat{F}_i \vee \ldots \vee \widehat{F}_j \vee \ldots \vee F_m\,.$$

For the map μ_*, we have the following result.

Proposition 2.1. *For $m \geq 1$, the map $\mu_*\colon S^mR \to S^{m-1}R$ is an intertwiner of $(R,\{\,\cdot\,,\,\cdot\,\})$-modules. Consequently, its dual $(\mu_*)^*\colon (S^{m-1}R)^* \to (S^mR)^*$ is an intertwiner too, and the induced map $\mu\colon \mathcal{C}^{n,m-1} \to \mathcal{C}^{n,m}$, defined as the dual of the map $\mathrm{Id}\otimes\mu_*\colon \Lambda^{n-m}R\otimes S^mR \to \Lambda^{n-m}R\otimes S^{m-1}R$, satisfies $\delta_{LA}\circ\mu = \mu\circ\delta_{LA}$.*

Proof. The first statement is a straightforward consequence of equation (2), and the remaining statements follow by dualisation and the fact that Lie algebra (co)homology is functorial in the representation. □

Note that μ_* neither squares to zero, nor commutes with α_*.

Proposition 2.2. *The commutator*

$$[\alpha_*,\mu_*]\colon \Lambda^{n-m}R \otimes S^mR \to \Lambda^{n-m+1}R \otimes S^{m-2}R$$

satisfies

$$[\alpha_*,\mu_*]u \otimes F_1 \vee \ldots \vee F_m = \sum_{i\neq j} u \wedge F_iF_j \otimes F_1 \vee \ldots \vee \widehat{F}_i \vee \ldots \vee \widehat{F}_j \vee \ldots F_m\,.$$

Proof. By polarisation, it suffices to check the case where all F_i are the same, say F. Then

$$\begin{aligned}
\mu_*\alpha_*(u\otimes F^{\vee m}) &= m(m-1)(m-2)u\wedge F\otimes F^2 \vee F^{\vee(m-3)}\\
\alpha_*\mu_*(u\otimes F^{\vee m}) &= m(m-1)(m-2)u\wedge F\otimes F^2 \vee F^{\vee(m-3)}\\
&\quad + m(m-1)u\wedge F^2 \otimes F^{\vee(m-2)}\,,
\end{aligned}$$

so the commutator is $m(m-1)u\wedge F^2\otimes F^{\vee(m-2)}$ as desired. □

Question 2.3. Is there something analogous to μ_* that behaves better with respect to α_*? One gets the feeling that the two structures (Lie algebra and commutative algebra) of R should be reflected in two differentials, a Lie algebra differential $\mathcal{C}^{n,m} \to \mathcal{C}^{n+1,m}$ in the 'antisymmetric direction' and a mirror image $\mathcal{C}^{n,m} \to \mathcal{C}^{n,m+1}$ in the 'symmetric direction' that consists of μ plus a part in which F_i somehow interacts with the $(R,\,\cdot\,)$-module $\Lambda^{n-m}R$. (This part is supposed to compensate for the nonzero commutator of μ and α.) Perhaps somehow related to Hochschild cohomology?

3 Nontrivial classes in $H^5(R,\mathbb{R})$ and $H^5(R,R^*)$

We assume that $(R,\{\cdot\,,\,\cdot\})$ is not perfect, and fix a nontrivial cocycle $\varepsilon\colon R\to\mathbb{K}$ with class $[\varepsilon]$ in $H^1(R,\mathbb{R})\simeq H^0(R,R^*)$.

Remark 1. The main example we have in mind is the following. Let (M,ω) be a symplectic manifold of dimension $2d$ and $R = C^\infty_c(M,\mathbb{R})$ with the usual multiplication and Poisson bracket. Then we have a nontrivial class $[\varepsilon]$ in $H^1(R,\mathbb{R})\simeq H^0(R,R^*)$ given by

$$\varepsilon(F) := \int_M F\omega^d\,.$$

This is (up to scaling) the only continuous class, because if ε is a distribution, then $\varepsilon(\{f,g\}) = 0$ implies $X_f\varepsilon = 0$ for all Hamiltonian vector fields X_f, so that ε is constant.

The map $\varepsilon\colon R\to\mathbb{K}$, considered as a 0-cocycle with values in R^*, yields a nontrivial cocycle $\mu^{m-1}\varepsilon$ in $C^0(R,S^mR^*)$. (In the context of locally convex Lie algebras, one should read S^mR^* as $(S^mR)'$, the continuous dual of S^mR.) We rescale it to $\psi^{0,m}$ by requiring

$$\psi^{0,m}(F_1\vee\ldots\vee F_m) = \varepsilon(F_1F_2\ldots F_m)\,.$$

We consider the 0-cocycles $\psi^{0,m}$, which reside in $C^0(R,S^mR^*)$, and chase them to the left in the following diagram, in which the horizontal lines are exact.

$$\begin{array}{ccccccccc}
 & & \uparrow\delta & & \uparrow\delta & & \uparrow\delta & & \\
0 & \longrightarrow & C^3(R,\mathbb{K}) & \xrightarrow{\alpha} & C^2(R,R^*) & \xrightarrow{\alpha} & C^1(R,S^2R^*) & \xrightarrow{\alpha} & \ddots \\
 & & \uparrow\delta & & \uparrow\delta & & \uparrow\delta & & \uparrow\delta \\
0 & \longrightarrow & C^2(R,\mathbb{K}) & \xrightarrow{\alpha} & C^1(R,R^*) & \xrightarrow{\alpha} & C^0(R,S^2R^*) & \xrightarrow{\alpha} & 0 \\
 & & \uparrow\delta & & \uparrow\delta & & \uparrow\delta & & \\
0 & \longrightarrow & C^1(R,\mathbb{K}) & \xrightarrow{\alpha} & C^0(R,R^*) & \xrightarrow{\alpha} & 0 & & \\
 & & \uparrow\delta & & \uparrow\delta & & & & \\
0 & \longrightarrow & C^0(R,\mathbb{K}) & \xrightarrow{\alpha} & 0 & & & & \\
 & & \uparrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

Proposition 3.1. *There exists a $k\in\{1,\ldots,m-1\}$ such that the cocycle $\psi^{0,m}$ gives rise to a nontrivial class in $H^{2(m-k)-1}(R,S^kR^*)$. (The case $k=0$ should be read as $H^{2m-1}(R,\mathbb{K})$).*

Proof. This is a standard diagram chase. Chasing $\psi^{0,m}$ to the left, we produce nonzero cochains $\psi^{1,m-1}$, $\psi^{3,m-2}$, $\psi^{5,m-3}$ etc. untill we reach one, say $\psi^{2r-1,m-r}$, which is closed. This happens for $r=m$ at the latest, because

$\alpha\delta\psi^{2m-1,0} = 0$ implies $\delta\psi^{2m-1,0} = 0$ due to the injectivity of α on $C^{2m-0}(\mathbb{R}, \mathbb{K})$. If $\psi^{2(m-k)-1,k}$ is exact, say $\delta\gamma^{2(m-k)-2,k}$, then we set $\tilde{\psi}^{2(m-k-1)-1,k+1} := \psi^{2(m-k+1)-1,k+1} - \alpha\gamma^{2(m-k)-2,k}$, and note that it is nonzero and closed. Continuing in this way, we find a k such that $\tilde{\psi}^{2(m-k)-1,k}$ is nonzero and closed but not exact. This happens for $k = 1$ at the latest, because $\tilde{\psi}^{1,m-1} = \delta\gamma^{0,m-1}$ would imply $\psi^{0,m} = \alpha\delta\gamma^{0,m-1} = \delta\alpha\gamma^{0,m-1} = 0$. $\square$

3.1 Classes induced from $\psi^{0,2}$ in $H^1(R, R^*)$ or $H^3(R, \mathbb{R})$

We assume that $R = C_c^\infty(M)$ for a symplectic manifold M of dimension $2d$, with $\varepsilon(f) = \int_M f\omega^d$. We define X_f by $i_{X_f}\omega = df$, so that $\{f, g\} = i_{X_f} i_{X_g}\omega$.

Starting from the invariant symmetric bilinear form $\psi^{0,2}(F_1 \vee F_2) = \varepsilon(F_1 F_2)$, we obtain, through $\psi^{1,1}(f)(F) = \varepsilon(fF)$, the cocycle $\psi^{3,0}(f_1 \wedge f_2 \wedge f_3) = \varepsilon(\{f_1, f_2\}f_3)$. (This is the analogue of the canonical class $\kappa(X, [Y, Z])$ for simple Lie algebras, derived from the invariant symmetric bilinear form κ.) The question is now whether $\psi^{3,0}$ is a coboundary.

3.1.1 Noncompact manifolds

Suppose that $\omega = d\theta$ is exact or, equivalently, that (M, ω) admits a vector field E such that $L_E\omega = \omega$. (In particular, this implies that M is not compact.) Then

$$[E, X_f] = X_{E(f)} - X_f\,, \tag{3}$$

because

$$d(E(f) - f) = L_E df - i_{X_f}\omega = L_E i_{X_f}\omega - i_{X_f} L_E\omega = i_{[E,X_f]}\omega\,.$$

We thus have

$$L_E\{f, g\} = \{L_E f, g\} + \{f, L_E g\} - \{f, g\}\,,$$

as

$$\begin{aligned} L_E\{f, g\} - \{f, L_E g\} - \{L_E f, g\} &= L_E X_f g - X_f L_E g - X_{E(f)} g \\ &= [E, X_f](g) - X_{E(f)} g \\ &= -X_f g. \end{aligned}$$

We define the cochain $\gamma^{2,0}$ by

$$\gamma^{2,0}(f_1 \wedge f_2) := \varepsilon(f_1 L_E f_2 + \tfrac{d}{2} f_1 f_2).$$

It is skew-symmetric because $L_E\omega^d = d\omega^d$, and $\delta\gamma^{2,0} = -\frac{d+2}{2}\psi^{3,0}$ because

$$\begin{aligned} \delta\gamma^{0,2}(f_1 \wedge f_2 \wedge f_3) &= \varepsilon(f_1 L_E\{f_2, f_3\} + \{f_1, f_3\} L_E f_2 - \{f_1, f_2\} L_E f_3) \\ &\quad + \tfrac{d}{2}\varepsilon(f_1\{f_2, f_3\} + \{f_1, f_3\} f_2 - \{f_1, f_2\} f_3) \\ &= \varepsilon(f_1 L_E\{f_2, f_3\}) - f_1\{L_E f_2, f_3\} - f_1\{f_2, L_E f_3\}) \\ &\quad - \tfrac{d}{2}\varepsilon(f_1\{f_2, f_3\}) \\ &= -\tfrac{d+2}{2}\psi^{3,0}(f_1 \wedge f_2 \wedge f_3)\,. \end{aligned}$$

we renormalise, $\Gamma^{2,0} = -\frac{2}{d+2}\gamma^{2,0}$, and obtain $\widetilde{\psi}^{1,1} := \psi^{1,1} - \alpha\Gamma^{2,0}$, namely

$$\widetilde{\psi}^{1,1}(f)(F) = \frac{2}{d+2}\int_M (f - L_E f) F\omega^d\,.$$

According to Proposition (3.1), $[\widetilde{\psi}^{1,1}]$ is a nontrivial class in $H^1(R, R^*)$, the first Lie algebra cohomology with values in the coadjoint representation.

Remark 2. If $R = C_c^\infty(X)$ where $X = M \times N$ with (M, ω) symplectic and $L_E\omega = \omega$, then one finds *coupled cocycles* in $H^2(R, \mathbb{R})$ [NW08]. It would be interesting to see whether this survives for more general Poisson manifolds.

3.1.2 compact manifolds

If M is compact, then certainly ω is not exact. In this case, we obtain a nontrivial class $[\psi^{3,0}]$ in $H^3(R, \mathbb{R})$.

Theorem 3.2. *Any derivation $D \colon R \to R^*$ such that $(f, F) \mapsto D(f)(F)$ is continuous is of the form $D(f) = S(df) + c\varepsilon(f)$ with $S \in \Omega_c^1(M)'$ a distributional vector field $S^\mu \partial_\mu$ with constant symplectic divergence $\partial_\mu S^\mu = c$.*

Proof. Joint w. Cornelia, in preparation. □

Theorem 3.3. *For M compact, the map that takes a singular k-chain into the corresponding distribution valued $n-k$ form in $C^{-\infty}(M, \Lambda^{n-k}T^*M) \simeq \Omega^k(M)'$ induces an isomorphism between singular homology in degree k and distribution valued De Rham cohomology in degree $n-k$.*

Proof. Cf. [Mel11]. □

Corollary 3.4. *Let M be compact and $\gamma \in \Omega^k(M)$ closed. If there exists a $\beta \in C^{-\infty}(M, \Lambda^{k-1}T^*M)$ such that $\gamma = d\beta$, then we also have $\gamma = d\beta'$ for some smooth β'.*

Proof. Every closed γ is cohomologous to the distribution valued k-form induced by its Poincaré dual. □

Proposition 3.5. *For M compact, the image of $\alpha \colon H^{1,1}(R, R^*) \to (S^2R^*)^R$ is zero.*

Proof. Since $\psi^{1,1}(f, F) = D(f)(F)$, we have $\alpha\psi^{1,1}(F_1, F_2) = D(F_1)(F_2) + D(F_2)(F_1)$. With $D(F) = S^\mu(\partial_\mu F)$, we have $\alpha\psi^{1,1}(F_1, F_2) = S^\mu(\partial_\mu(F_1F_2))$, which equals $c\int_M F_1F_2\omega^d$. For compact M, the image $\mathrm{Im}(\alpha) \subseteq (S^2R^*)^R$ is zero, because a distributional vector field S with symplectic divergence c corresponds with a distributional 1-form γ with $d\gamma = c\omega$, but these only exist for $c = 0$ by the previous Corollary. □

Corollary 3.6. *The map $(S^2R^*)^R \to H^{3,0}(R, \mathbb{R})$ is injective. In particular, the class $[\psi^{3,0}] \in H^3(R, \mathbb{R})$ is nontrivial.*

Proof. This follows from Prop 3.5 and the five term exact sequence of Prop. 1.3. □

3.2 Classes induced from $\psi^{0,3}$ in $H^5(R, \mathbb{K})$, $H^3(R, R^*)$ or $H^1(R, S^2R^*)$

Here I can still do the diagram chase to the left, but then going to the right is hard because you have to find preimages of δ rather than α, or prove that they don't exist.

3.3 Classes induced from $\psi^{0,5}$ in $H^7(R, \mathbb{K})$ or $H^5(R, R^*)$

After this warmup exercise, we try to determine which classes are generated by $\psi^{0,4}$. This could be interesting because we know that there is an exceptional class in $H^7(R, \mathbb{R})$ [GKF72]. (Also, there are exceptional classes in $H^9(R, \mathbb{R})$ (Metoki) and higher (Mikami-Nakae-Kodama), which you can try to hit with $\psi^{0,5}$ and higher. Apparently these are useful for studying transversely symplectic foliations (Kotschick-Morita).) If $\psi^{0,4}$ doesn't hit the Gelfand-Kalinin-Fuks class, then another candidate is Dzhumadil'daev's class in $H^5(R, R)$ [Dzh04].

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