Localisation of Lie Algebra Cohomology

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Abstract

Some loose thoughts as to how and when the cohomology of a cosheaf of Lie algebra can be ascertained locally. Beware! This is by no means a preprint: the proofs are in various degrees of incompleteness, and statements should not be trusted blindly. Please don't distribute.

1 Precosheaves of Lie algebras

Let X be a topological space and let $\mathcal{O}(X)$ be the collection of open sets, ordered by inclusion. A precosheaf of Lie algebras is a functor L from $\mathcal{O}(X)$ to the category of Lie algebras: for each open set, we have a Lie algebra $L(U)$, for each inclusion $V \subset U$ we have a Lie algebra homomorphism $\iota_{UV} : L(V) \to L(U)$, the trivial inclusion $U \subseteq U$ corresponds to the identity $\iota_{UU} = \mathbb{I}$ on $L(U)$, and $W \subseteq V \subseteq U$ implies $\iota_{UV}\iota_{VW} = \iota_{UW}$.

A presheaf of vector spaces is a contravariant functor R from $\mathcal{O}(X)$ to the category of vector spaces: for each open set U we have a vector space $R(U)$, for each inclusion $V \subset U$ we have a linear map $J_{UV} : R(U) \to R(V)$, the inclusion $U \subseteq U$ yields the identity $J_{UU} = \mathbb{I}$, and $W \subseteq V \subseteq U$ implies $J_{WV}J_{VU} = J_{WU}$. A presheaf of representations is a presheaf of vector spaces where each $R(U)$ carries a representation π_U of $L(U)$, compatible in the sense that

$$J_{VU} \cdot \pi_U \circ \iota_{UV} = \pi_V \cdot J_{VU} \,.$$

1.1 Precosheaves of cohomologies

We denote by $C^{\bullet}(L, R)$ the cochain complex of alternating multilinear maps $\psi : L^n \to R$ with differential $\delta : C^n(L, R) \to C^{n+1}(L, R)$ given by

$$\begin{aligned} \delta\psi(X_0, \ldots, X_n) \;&:=\; \sum_{k=0}^{n} (-1)^k X_k \cdot \psi(X_0, \ldots, \hat{X}_k, \ldots, X_n) \\ &\quad + \sum_{0 \leq k < l \leq n} (-1)^{k+l} \psi([X_k, X_l], X_0, \ldots, \hat{X}_k, \ldots, \hat{X}_l, \ldots, X_n) \,. \end{aligned}$$

Proposition 1.1 *Let L be a precosheaf of Lie algebras, and let R be a presheaf of representations. Then for each $n \in \mathbb{N}$, the assignment $U \mapsto C^n(L(U), R(U))$ constitutes a presheaf of vector spaces, and δ is a morphism of presheaves. In particular, the assignment $U \mapsto H^n(L(U), R(U))$ constitutes a presheaf of vector spaces.*

Proof : If $V \subseteq U$, then the Lie algebra homomorphism $\iota_{UV} : L(V) \to L(U)$ induces a chain map $\iota^* : C^\bullet(L(U), R(U)) \to C^\bullet(L(V), R(V))$ by $(\iota^*\psi)(X_1, \ldots, X_n) := J_{VU}\psi(\iota_{UV}(X_1), \ldots, \iota_{UV}(X_n))$. Indeed, for any n-cochain ψ we have

$$\begin{aligned}
\iota^*\delta\psi(X_0, \ldots, X_n) &= \sum_{0\leq i<j\leq n} (-1)^{i+j} J_{VU}\psi([\iota_{UV}(X_i), \iota_{UV}(X_j)], \iota_{UV}(X_0), \ldots, \hat{\imath}, \ldots, \hat{\jmath}, \ldots, \iota_{UV}(X_n)) \\
&\quad + \sum_{0\leq k\leq n} (-1)^k J_{VU}\pi_U(\iota_{UV}(X_k))\psi(\iota_{UV}(X_0), \ldots, \hat{k}, \ldots, \iota_{UV}(X_n)) \\
&= \sum_{0\leq i<j\leq n} (-1)^{i+j}\psi(\iota_{UV}([X_i, X_j], \iota_{UV}(X_0), \ldots, \hat{\imath}, \ldots, \hat{\jmath}, \ldots, \iota_{UV}(X_n)) \\
&\quad + \sum_{0\leq k\leq n} (-1)^k \pi_V(X_k) J_{VU}\psi(\iota_{UV}(X_0), \ldots, \hat{k}, \ldots, \iota_{UV}(X_n)) \\
&= \delta\,\iota^*\,\psi(X_0, \ldots, X_n)
\end{aligned}$$

We therefore have restriction maps $\rho_{VU} : H^n(L(U), R(U)) \to H^n(L(V), R(V))$ satisfying the presheaf property $\rho_{WV} \circ \rho_{VU} = \rho_{WU}$. □

Remark 1.2 *Note that the cohomology is* not *automatically a sheaf if L is a cosheaf and R a sheaf. Take for example the (flabby) cosheaf of Lie algebras $L(U) = C_c^\infty(U)$ with the trivial bracket. The second (continuous) cohomology with trivial coefficients $H^2(L(U), \mathbb{R})$ is simply the space of (continuous) skew-linear maps $\psi : C_c^\infty(U) \times C_c^\infty(U) \to \mathbb{R}$. This is a presheaf, but not a sheaf. The problem here is not gluing, but local identity. If X is covered by U_1 and U_2, and $\psi_{1,2}$ on $L(U_{1,2})$ are given by, say, $\psi_1(f, g) := \int_{U_1\times U_1} f(x)\kappa_1(x, y)g(y)dxdy$ and $\psi_2(f, g) := \int_{U_2\times U_2} f(x)\kappa_2(x, y)g(y)dxdy$, (we assume X to be a manifold equipped with a volume form dx) then if $\psi_1|_{U_1\cap U_2} = \psi_2|_{U_1\cap U_2}$, we have $\kappa_1(x, y) = \kappa_2(x, y)$ if $x, y \in U_1 \cap U_2$. We can therefore extend κ_1 and κ_2 to a kernel κ on $X \times X$, so that the gluing axiom is fulfilled. But this extension is highly non-unique; the 'diagonal' terms $\kappa|_{U_1\times U_1}$ and $\kappa|_{U_2\times U_2}$ are of course determined by κ_1 and κ_2, but the 'off-diagonal' terms $\kappa_{(U_1/U_2)\times(U_2/U_1)}$ can be specified more or less at will. There is no hope of satisfying the 'local identity'. If it is a sheaf at all, it will be a sheaf over $X \times X/S_2$, not over X.*

1.1.1 The Precosheaf over the Spectrum

Given a Lie algebra L, one can obtain a topological space and a precosheaf of Lie algebras in the following fashion.

Definition 1.3 *An ideal P of a Lie algebra L is called* prime *if for any two ideals I and J, $[I, J] \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$. The* prime spectrum $\mathrm{Spec}^p(L)$ *is defined as the set of all proper prime ideals $P < L$.*

We endow the prime spectrum with the 'Zariski topology' in the usual fashion: we declare the closed sets to be those of the form

$$V(I) := \{P \in \mathrm{Spec}^p(L) \mid P \supseteq I\}$$

with I is an ideal in L. We denote the complementary open sets by $U(I) := \mathrm{Spec}^p(L) - V(I)$, and we denote by $I^\circ := \bigcap_{P\in V(I)} P$ the biggest ideal J such that $V(J) = V(I)$.

Proposition 1.4 *This makes* $\mathrm{Spec}^p(L)$ *into a topological space. The locale of open sets is isomorphic to the locale of 'open' ideals* $I = I^\circ$*, equipped with the operations* $I \vee J := I + J$ *and* $I \wedge J := I \cap J = [I, J]^\circ$*.*

Proof : We have a 1 : 1-correspondence between 'open' ideals I° and open sets $U(I^\circ)$.

- Both $\emptyset = U(\{0\}) = V(L)$ and $\mathrm{Spec}^p(L) = U(L) = V(\{0\})$ are open as well as closed.
- $\bigcap_{\alpha\in A} V(I_\alpha) = V(\sum_{\alpha\in A} I_\alpha)$, where $\sum_{\alpha\in A} I_\alpha$ is the ideal of finite sums of elements of I_α. Therefore, $\bigcup_{\alpha\in A} U(I_\alpha) = U(\sum_{\alpha\in A} I_\alpha)$. In particular, arbitrary intersections of closed sets are closed and arbitrary unions of open sets are open.
- $V(I_1) \cup V(I_2) = V([I_1, I_2])$ because for any prime ideal P, $P \supset I_1$ or $P \supset I_2$ is equivalent to $P \supset [I_1, I_2]$. Finite unions of closed sets are thus closed and finite intersections of open sets are open: $U(I_1) \cap U(I_2) = U([I_1, I_2])$.

This shows that $\mathrm{Spec}^p(L)$ is a topological space, that unions of open sets correspond to sums of ideals and that intersections of open sets correspond to commutators of ideals.

Finally, $P \supset I^\circ \cap J^\circ$ implies $P \supset I \cap J$, which implies $P \supset [I, J]$, which implies $P \supset I$ or $P \supset J$, which implies $P \supset I^\circ$ or $P \supset J^\circ$ which implies $P \supset I^\circ \cap J^\circ$. Thus $I^\circ \cap J^\circ = (I \cap J)^\circ = [I, J]^\circ$. □

The closure of $U \subseteq \mathrm{Spec}^p(L)$ is given by $\overline{U} = \{Q \in \mathrm{Spec}^p(L) \mid Q \supseteq \bigcap_{P\in U} P\}$, or $\overline{U} = V(\bigcap_{P\in U} P)$. Indeed, the smallest closed set containing U corresponds to the biggest ideal I such that $P \supseteq I$ for all $P \in U$, which is obviously $\bigcap_{P\in U} P$.

We've already used that intersections and unions of sets correspond to sums and commutators of ideals. This correspondence is a functor.

Proposition 1.5 *The prime spectrum is a covariant functor from the category of Lie algebras to the category of locales,* $\mathrm{Spec}^p : \mathbf{Lie} \to \mathbf{Loc}$.

Proof : As a locale, the topological space $\mathrm{Spec}^p(L)$ is isomorphic to the set of 'open' ideals in L with $I \vee J := I + J$ and $I \wedge J := [I, J]^\circ$. (We call an ideal I in L 'open' if $I = I^\circ$.) If $\phi : L \to L'$ is a homomorphism of Lie algebras, then $\phi^{-1} : I' \mapsto \phi^{-1}(I')$ maps open sets in $\mathrm{Spec}^p(L')$ to open sets in $\mathrm{Spec}^p(L)$ in a way that preserves $\wedge$ and $\vee$, i.e. it is a morphism of frames. Since the category of locales is the opposite category of the category of frames, every homomorphism $\phi : L \to L'$ defines a morphism of locales $\mathrm{Spec}^p(\phi) : \mathrm{Spec}^p(L) \to \mathrm{Spec}^p(L')$. □

Note that although the locales $\mathrm{Spec}^p(L)$ are honest topological spaces and the $\mathrm{Spec}^p(\phi)$ are morphisms of locales, they need not be induced by continuous maps because the inverse image of a prime ideal need not be prime. The situation is different from that in commutative rings, where the spectrum is a contravariant functor to the category of topological spaces because inverse images of prime ideals of commutative algebras are prime.

The closed points in $\mathrm{Spec}^p(L)$ are exactly the maximal ideals. The following Lemma therefore shows that if L is perfect, then all points are closed (and, in particular, $\mathrm{Spec}^p(L)$ is a T^1-space).

Proposition 1.6 *If* L *is perfect,* $[L, L] = L$*, then every maximal ideal is prime.*

Proof : Let M be a maximal ideal, and let $I \cap J \subseteq M$ for two ideals I and J. Suppose that neither one is contained in M. By maximality of M, we then have $I + M = L$ and $J + M = L$. Thus $[L, L] = [I, J] + [I, M] + [M, J] + [M, M] \subseteq M$, contradicting the fact that L is perfect. $\square$

Definition 1.7 *We call the set* $\mathrm{Spec}^m(L)$ *of maximal ideals of* L *the* maximal ideal spectrum. *If* L *is perfect, then* $\mathrm{Spec}^m(L)$ *inherits the subspace topology from* $\mathrm{Spec}^p(L)$. *The closure of* $U \subseteq \mathrm{Spec}^m(L)$ *with respect to this topology is* $\overline{U} = \{M \in \mathrm{Spec}^m(L)\,;\, \bigcap_{Q \in U} Q \subseteq M\}$.

For each Lie algebra L, we thus obtain a (flabby) precosheaf of Lie algebras over $\mathrm{Spec}^p(L)$ by setting $L(U) := \bigcap_{Q \in U^c} Q$ for each open U, and if L is perfect, we can do the same with $\mathrm{Spec}^m(L)$.

Remark 1.8 *In this level of generality, I do not believe there is sufficient control over the precosheaves of Lie algebras obtained in this fashion to reach any localisation results on the cohomology. I'm just stating this because it serves as motivation, and because in many examples of cosheaves of Lie algebras (e.g. the cosheaf of compactly supported vector fields), the base space can be recovered from the global sections in the manner here described. This is Pursell-shanks' theorem [SP54], which holds in great generality.*

1.2 Full cohomology vs. local cohomology

We define the *local* cohomology of a precosheaf of Lie algebras.

Definition 1.9 *A collection* $\{U_1, \ldots, U_n\}$ *of sets is called* connected *if for any* $1 \leq i, j \leq n$, *there exist* $i = i_1, i_2, \ldots, i_{k-1}, i_k = j$ *such that* $U_{i_s} \cap U_{i_{s+1}} \neq \emptyset$ *for all* $1 \leq s \leq k-1$.

This is not quite equivalent to $\bigcup_{i=1}^n U_i$ being connected, because the U_i are allowed to be empty or disconnected. (I'm not sure which one of the two is the proper definition.)

Definition 1.10 *A cochain* $\psi \in C^n(L(U), R(U))$ *is called* local *if* $\rho_{U_0 U}\psi(X_1, \ldots, X_n) = 0$ *for all* $X_i \in \iota_{UU_i}(L(U_i))$, $i = 1, \ldots, n$, *such that* $\{U_0, U_1, \ldots, U_N\}$ *is not connected. The vector space of local cochains is denoted* $C^n_{\mathrm{loc}}(L(U), R(U))$.

Note that for $\mathbb{R}$ the constant sheaf $R(U) = \mathbb{R}$ with values in the trivial representation, this reduces to '$\psi(X_1, \ldots, X_n) = 0$ for all $X_i \in \iota_{UU_i}(L(U_i))$ such that $\{U_1, \ldots, U_N\}$ is not connected'.

Also note that any collection containing $\emptyset$ is disconnected. Consequently, a local cochain ψ satisfies $\psi(X_1, \ldots, X_n) = 0$ as soon as *any* of the X_i is in $\iota_{U\emptyset}L(\emptyset)$. The above notion of 'locality' is a weaker condition than being *diagonal* in the sense of Losev, because the latter requires ψ to vanish if $\cap_{i=1}^{n-1} U_i = \emptyset$.

Note that $U \mapsto C^n_{\mathrm{loc}}(L(U), R(U))$ is a sub-precosheaf of $U \mapsto C^n(L(U), R(U))$. The differential δ restricts to map of presheaves $C^n_{\mathrm{loc}}(L, R) \to C^{n+1}_{\mathrm{loc}}(L, R)$, because $\{[U_i, U_j], \ldots, \hat{U}_i, \ldots, \hat{U}_j, \ldots U_n\}$ is automatically disconnected whenever $\{U_0 \ldots U_n\}$ is. We therefore have a natural map

$$H^n_{\mathrm{loc}}(L, R) \to H^n(L, R)\,.$$

The aim is to prove that the algebra $H^*(L,R)$ is generated by the image of $H^n_{\mathrm{loc}}(L,R)$. In case of continuous cocycles on a precosheaf of locally convex topological Lie algebras, the proper statement is of course that the algebra generated by the image of $H^*_{\mathrm{loc}}(L,R)$ is dense in $H^*(L,R)$.

1.2.1 The local cohomology generates the full cohomology

Let L be a precosheaf of Lie algebras such that $[\iota_{XU}L(U), \iota_{XV}L(V)] = 0$ if $U \cap V = \emptyset$. For brevity, write $L_X(U)$ for $\iota_{XU}L(U)$.

Lie algebra cohomology, with chains $C^n(L(X),\mathbb{R})$ and differential δ is dual to Lie algebra homology with chains $C_n(L(X),\mathbb{R}) = \wedge^n L(X)$ and differential $D : C_n \to C_{n-1}$ given by $D(\wedge_{i=1}^n X_i) = \sum_{1\leq i<j\leq n}(-1)^{i+j}[X_i,X_j]\wedge_{s\neq i,j} X_s$. It is readily verified that $U \mapsto C_\bullet(L(U),\mathbb{R})$ is a precosheaf, and that $D^k|_U \circ \iota_{UV} = \iota_{UV} \circ D^k|_V$. (We write $D^k|_U$ for the restriction of D^k to $\wedge^k L(U)$, and $D^k|_{X,U}$ for the restriction of D^k_X to $L_X(U)$.)

The key observation in the following is that $C_\bullet(L(X),\mathbb{R})$ is a (supercommutative graded) algebra, and if $[L_X(U), L_X(V)] = 0$ with $X \in C_m(L_X(U),\mathbb{R})$ and $Y \in C_{m'}(L_X(V),\mathbb{R})$, then

$$D(X \wedge Y) = D(X)\wedge Y + (-1)^{\deg(X)} X \wedge D(Y)$$

because all terms mixing $L_X(U)$ and $L_X(V)$ vanish.

Lemma 1.11 *Let L be a precosheaf of Lie algebras satisfying $[L_X(U), L_X(V)] = 0$ if $U \cap V = \emptyset$. Every cocycle ψ^n is cohomologous to a cocycle $\tilde{\psi}^n$ such that $\tilde{\psi}^n(X \wedge DY) = 0$ and $\tilde{\psi}^n(DX \wedge Y) = 0$ for all $X \in \wedge^k \iota_{XU}(L(U))$ and $Y \in \wedge^{n-k+1}\iota_{XV}(L(V))$ such that $U \cap V = \emptyset$ and $k = 0,\ldots,n+1$. If L is a precosheaf of locally convex topological Lie algebras, $\tilde{\psi}$ can be chosen to be continuous if ψ is.*

Proof : If ψ^n is a cocycle on $L(X)$, and U, V are disjoint open subsets of X, then define

$$\begin{aligned} \gamma^{n-1} \quad &: \quad D(\wedge^{k+1}L_X(U)) \times D(\wedge^{n-k+1}L_X(V)) \to \mathbb{R} \\ (DX, DY) \quad &\mapsto \quad \psi^n(X \wedge DY)\,. \end{aligned}$$

This is well defined. Suppose that $DX = DX'$. Then $\psi^n(X \wedge DY) = \psi^n(X' \wedge DY)$, because

$$\delta\psi^n(X\wedge Y) = \psi^n(DX \wedge Y) + (-1)^{\deg(X)}\psi^n(X \wedge DY) = 0$$

implies $D(X - X') = 0 \Rightarrow \psi^n(X - X' \wedge DY) = 0$. This also shows that γ can be equivalently defined as $\gamma^{n-1}(DX, DY) = (-1)^{\deg(X)+1}\psi^n(DX, Y)$.

GAP: THIS ALSO SHOWS THAT γ IS SEPARATELY CONTINUOUS IF $L(U)$ IS A LOCALLY CONVEX LIE ALGEBRA AND ψ IS CONTINUOUS. WE NEED THAT IT IS JOINTLY CONTINUOUS IN THAT CASE, I DON'T SEE WHY IT SHOULD BE.

Because $\gamma^{n-1} : D(\wedge^{k+1}L_X(U)) \times D(\wedge^{n-k+1}L_X(V)) \to \mathbb{R}$ is bilinear, it defines a linear map $\gamma^{n-1} : D(\wedge^{k+1}L_X(U)) \otimes D(\wedge^{n-k+1}L_X(V)) \to \mathbb{R}$, and thus a linear map $\gamma^{n-1} : D(\wedge^{k+1} L_X(U)) \wedge D(\wedge^{n-k+1} L_X(V)) \to \mathbb{R}$. Our definition of γ^{n-1} depends on k, U and V.

We wish to show that the different versions $\gamma^{n-1}_{k,U,V}$ agree on the overlap of their domains, so that a single γ^{n-1} on

$$\mathrm{Span}\langle DX \wedge DY\, ;\, X \in \wedge^{k+1} L_X(U), Y \in \wedge^{n-k+1} L_X(V), U \cap V = \emptyset, k = 0, \ldots, n+1\rangle$$

is well defined. We need to show that if $DX \wedge DY = DX' \wedge DY'$, then $X \wedge DY - X' \wedge DY'$ is in the image of D.

THERE'S A GAP HERE. PROBABLY USE THE COSHEAF PROPERTY OF L, OR PERHAPS TRY TO PROVE THAT $\wedge^n L$ IS A COSHEAF OVER THE SYMMETRIC PRODUCT X^n/S_n. WE'LL ASSUME THAT THE VARIOUS $\gamma^{n-1}_{k,U,V}$ ARE COMPATIBLE.

Then extend γ^{n-1} from this linear span to a cocycle Γ^{n-1} on $\wedge^n L(X)$. If γ^{n-1} is continuous, one can choose Γ^{n-1} to be continuous by the Hahn-Banach theorem for locally convex topological vector spaces.

Then for $X \in \wedge^k L_X(U)$, $Y \in \wedge^{n-k} L_X(V)$, one has

$$\delta\Gamma^{n-1}(X \wedge Y) = \Gamma^{n-1}(DX \wedge Y + (-1)^{\deg(X)} X \wedge DY)\,,$$

which equals $\psi^n(X \wedge Y)$ if either $Y \in D(\wedge^{n-k+1}(L_X(V)))$ or $X \in D(\wedge^{k+1}(L_X(U)))$.

If we define[1] $\tilde{\psi}^n := \psi^n - \delta\Gamma^{n-1}$, then $\tilde{\psi}^n$ vanishes on $D(\wedge^{k+1} L_X(U)) \times \wedge^{n-k} L(V)$ and on $\wedge^k L(U) \times D(\wedge^{n-k+1} L_X(V))$ for all open disjoint $U, V \subseteq X$. In other words, for $X \in \wedge^k L(U)$ and $Y \in \wedge^{n-k} L(V)$, we have not only $\tilde{\psi}^n(DX \wedge Y + (-1)^k X \wedge DY) = 0$, but we even have

$$\tilde{\psi}^n(DX \wedge Y) = 0 \qquad \text{and} \qquad \tilde{\psi}^n(X \wedge DY) = 0$$

separately. □

Theorem 1.12 (Conjectural!) *Let L be a precosheaf of nuclear topological Lie algebras satisfying $[L_X(U), L_X(V)] = 0$ if $U \cap V = \emptyset$. Then the algebra generated by the local cohomology $H^*_{\mathrm{loc}}(L(X), \mathbb{R})$ is dense in $H^*(L(X), \mathbb{R})$.*

Proof : If $\tilde{\psi}$ is continuous, and vanishes on $\wedge^k L(U) \wedge D(\wedge^{n-k+1} L_X(V))$ and on $D(\wedge^{k+1} L_X(U)) \wedge \wedge^{n-k} L_X(V)$ for all $U \cap V = \emptyset$, then it defines a continuous linear functional $\tilde{\psi}_{U,V}$ on

$$\wedge^k L_X(U)/\overline{D(\wedge^{k+1} L_X(U))}\ \overline{\otimes}\ \wedge^{n-k} L_X(V)/\overline{D(\wedge^{n-k+1} L_X(V))} \qquad (1)$$

for all disjoint $U, V \subseteq X$. (The $\overline{\otimes}$ denotes the closure of the tensor product w.r.t. the topology induced by the inclusion into $\wedge^n L(X)$.) Now $\wedge^k L_X(U)/\overline{D(\wedge^{k+1} L_X(U))}$ is a subspace of $\wedge^k L(X)/\overline{D(\wedge^{k+1} L(X))}$ and similarly $\wedge^{n-k} L_X(V)/\overline{D(\wedge^{n-k+1} L_X(V))}$ is a subspace of $\wedge^{n-k} L(X)/\overline{D(\wedge^{n-k+1} L(X))}$. Since the $\tilde{\psi}_{U,V}$ are compatible for different pairs U, V,

[1] Note that if $\psi^n = \delta\chi^{n-1}$, then on $\mathrm{Im}D^{k+1}|_U \times \mathrm{Im}D^{n-k}|_V$, we have $\gamma^{n-1} = \chi^{n-1}$. Thus $\tilde{\psi}^n = \delta(\tilde{\chi}^{n-1})$, where $\tilde{\chi}^{n-1} := \chi^{n-1} - \Gamma^{n-1}$ vanishes on $\mathrm{Im}D^{k+1}|_U \times \mathrm{Im}D^{n-k}|_V$.

THIS IS PRECISELY THE PART WE STILL NEED TO PROVE!

this defines a continuous linear functional on the subspace of

$$\wedge^k L(X)/\overline{D(\wedge^{k+1}L(X))} \,\hat{\otimes}\, \wedge^{n-k} L_X(V)/\overline{D(\wedge^{n-k+1}L_X(V))} \tag{2}$$

generated by the spaces (1). Use the Hahn-Banach theorem to extend this to a continuous linear functional on (2). That (1) is a subspace of 2 requires the assumption that $\overline{\otimes}$ and $\hat{\otimes}$ are compatible. In order to assure this, we assume that L is a precosheaf of *nuclear spaces*. Subspaces, tensor products and quotients by closed subspaces of nuclear spaces are again nuclear, and $(E\hat{\otimes}F)' \simeq E'\hat{\otimes}F'$ [Gro52]. Since $\big(\wedge^k L(X)/\overline{D(\wedge^{k+1}L(X))}\big)'$ is exactly the space of continuous *closed* k-cochains on $L(X)$, we obtain *closed* k- and $n-k$-cochains ϕ^k_α and ϕ^{n-k}_α on $L(X)$ such that the induced element on (2) can be written $\sum_{\alpha=1}^{\infty}\phi^k_\alpha \otimes \phi^{n-k}_\alpha$. If we now consider the induced element $\sum_{\alpha=1}^{\infty}\phi^k_\alpha \wedge \phi^{n-k}_\alpha$ in $\wedge^n L(X)$, then it coincides with $\tilde{\psi}$ on $\wedge^k L_X(U) \wedge \wedge^{n-k} L_X(V)$ if U and V are disjoint. That is to say: $\hat{\psi} := \tilde{\psi} - \sum_{\alpha=1}^{\infty}\phi^k_\alpha \wedge \phi^{n-k}_\alpha$ vanishes on $\wedge^k L_X(U) \wedge \wedge^{n-k} L_X(V)$ for all disjoint U and V.

If we repeat this procedure for $k = 1, \ldots, n$, in each step respecting the above property for all the k's you've already handled,

HOW???

then the resulting cocycle ψ_{loc} vanishes on $\wedge^k L_X(U)\wedge\wedge^{n-k} L_X(V)$ for all disjoint U and V and for all k, and is therefore local. We see that every continuous cocycle ψ on $L(X)$ is cohomologous to $\psi_{\mathrm{loc}} + \sum_{k=1}^{n-1}\sum_{\alpha=1}^{\infty}\phi^k_\alpha \wedge \phi^{n-k}_\alpha$, the sum of a local cocycle and a term generated by (possibly nonlocal) cocycles of smaller degree.

The statement now follows by induction: certainly, the first Lie algebra cohomology is generated by the local cohomology. (It is local itself.) Suppose that the cohomology up to and including degree $n-1$ is generated by the local cohomology. Then if ψ is any n-cocycle, both ψ_{loc} and the ϕ^k_α are generated by local cocycles. This means that also ψ is generated by local cocycles. □

Remark 1.13 *Apart from the holes in the proof, we also haven't shown that the map* $H^n_{\mathrm{loc}}(L,R) \to H^n(L,R)$ *is injective. This is something you would certainly like to have.*

1.3 A double complex

If $\mathcal{U} = \{U_i\,;\, i \in I\}$ is a cover of X, then denote by $\check{C}^n C^m(L,R,\mathcal{U})$ the space of Čech cocycles w.r.t. $\mathcal{U}$. If we define $U_{i_0,\ldots,i_n} := U_{i_0} \cap \ldots U_{i_n}$, then a Čech n-cocycle $\psi_\bullet$ assigns to each tuple $(i_0,\ldots,i_n)$ a Lie m-cocycle $\psi_{i_0,\ldots,i_n} \in C^m(L(U_{i_0,\ldots,i_n}), R(U_{i_0,\ldots,i_n}))$, in such a way that $\psi_{i_{\sigma(0)}}, \ldots, i_{\sigma(n)} = (-1)^{\mathrm{sg}(\sigma)}\psi_{i_0,\ldots,i_n}$.

Remark 1.14 *Although strictly speaking everything in this section ought to make sense for the precosheaf of arbitrary cochains* $U \mapsto C^n(L(U), R(U))$*, the assumptions we will need (especially regarding acyclicity of the presheaf of cochains) will make sense only in the context of* local *cohomology. Everywhere where it says 'cochain', one should keep in mind 'continuous local cochain'.*

The Lie algebra differential $\delta : \check{C}^n C^m(L,R,\mathcal{U}) \to \check{C}^n C^{m+1}(L,R,\mathcal{U})$ commutes with the Čech differential $d : \check{C}^n C^m(L,R,\mathcal{U}) \to \check{C}^{n+1} C^m(L,R,\mathcal{U})$, so we obtain the following double complex.

$$\begin{array}{ccccccccccc}
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \delta\big| & & \delta\big| & & \delta\big| & & \delta\big| & & \\
0 & \longrightarrow & C^2(L(M),R(M)) & \xrightarrow{\rho} & \check{C}^0C^2(L,R,\mathcal{U}) & \xrightarrow{d} & \check{C}^1C^2(L,R,\mathcal{U}) & \xrightarrow{d} & \check{C}^2C^2(L,R,\mathcal{U}) & \xrightarrow{d} & \\
 & & \delta\big\uparrow & & \delta\big\uparrow & & \delta\big\uparrow & & \delta\big\uparrow & & \\
0 & \longrightarrow & C^1(L(M),R(M)) & \xrightarrow{-\rho} & \check{C}^0C^1(L,R,\mathcal{U}) & \xrightarrow{-d} & \check{C}^1C^1(L,R,\mathcal{U}) & \xrightarrow{-d} & \check{C}^2C^1(L,R,\mathcal{U}) & \xrightarrow{-d} & \\
 & & \delta\big\uparrow & & \delta\big\uparrow & & \delta\big\uparrow & & \delta\big\uparrow & & \\
0 & \longrightarrow & R(M) & \xrightarrow{\rho} & \check{C}^0R & \xrightarrow{d} & \check{C}^1R & \xrightarrow{d} & \check{C}^2R & \xrightarrow{d} & \\
 & & \big\uparrow & & \big\uparrow & & \big\uparrow & & \big\uparrow & & \\
 & & 0 & & 0 & & 0 & & 0 & &
\end{array}$$

The occasional minus signs are merely a matter of convention; they make sure that the two differentials in the complex anticommute rather than commute.

Note that $C^0(L,R,\mathcal{U}) = R$, so that the kernel of $\delta : R(U) \to C^1(L(U),R(U))$ is $\mathrm{Ann}(L(U)) := \{r \in R(U); \pi_U(X)r = 0 \,\forall\, X \in L(U)\}$ by definition. The homology of the n^{th} column $(\check{C}^n C^\bullet(L,R,\mathcal{U}),\delta)$ therefore calculates the Lie algebra cohomology $H^\bullet(L(U_{i_0,\ldots,i_n}),R(U_{i_0,\ldots,i_n}))$ on the intersections of the sets in the cover, with the convention that $H^0(L(U),R(U)) = \mathrm{Ann}(L(U))$, the annihilator $\mathrm{Ann}(L(U)) := \{r \in R(U)\,;\, \pi_U(X)r = 0 \,\forall\, X \in L(U)\}$. We are interested in the homology of the 0^{th} column, $H^\bullet(L(M)),R(M))$.

Note also that the homology of the m^{th} row, $H^\bullet(\check{C}^\bullet C^m(L,R,\mathcal{U}),d)$, is zero at the first spot for all $\mathcal{U}$, (i.e. $H^{-1}(\check{C}^\bullet C^m(L,R,\mathcal{U}),d) = \{0\}$ for all $\mathcal{U}$) if and only if the presheaf of Lie cochains $C^m(L,R)$ satisfies the local identity axiom. Its cohomology at the second spot is zero for all $\mathcal{U}$, ($H^0(\check{C}^\bullet C^m(L,R,\mathcal{U}),d) = \{0\}$ for all $\mathcal{U}$), if and only if the presheaf $C^m(L,R)$ satisfies the gluing axiom. Thus the presheaf of Lie cochains is a sheaf precisely if $H^{-1}(\check{C}^\bullet C^m(L,R,\mathcal{U}),d) = H^0(\check{C}^\bullet C^m(L,R,\mathcal{U}),d) = \{0\}$ for all $\mathcal{U}$.

We try to extract information from this complex by using the two spectral sequences that converge to the diagonal complex. We specialise to the case $R = \mathbb{R}$. Although the bottom nonzero row of the sequence then induces $E^{p,0}_\infty = H^p(M,\mathbb{R})$ for $p \geq 0$, $E^{-1,0}_\infty = \mathbb{R}$, these terms are not connected to the rest of the diagram because $\delta : \check{C}^n\mathbb{R} \to \check{C}^n C^1(L,\mathbb{R},\mathcal{U})$ is simply the zero map. In the remainder, we therefore set this bottom row to zero.

The cohomology of the 'total' complex can now be calculated in two different ways: by a spectral sequence ${}_IE^{\bullet,\bullet}_r$ with second page ${}_IE^{p,q}_2 = H^p(H^q(\check{C}^{p'}C^{q'},\delta),(-1)^{q'}d)$, and by a spectral sequence ${}_{II}E^{\bullet,\bullet}_r$ with second page ${}_{II}E^{p,q}_2 = H^p(H^q(\check{C}^{p'}C^{q'},(-1)^{q'}d),\delta)$.

Assume that, for some reason, we knew that the presheaves $C^q(L,\mathbb{R})$ were in fact acyclic sheaves. (We will show that something like this happens for the sheaves of *local* cochains if L is sufficiently 'soft'.) Then ${}_{II}E^{p,q}_2 = 0$, because the cohomology $H^p(\check{C}^\bullet C^q, d)$ vanishes. (I mean the cohomology w.r.t. d of each row in the above complex; vanishing for $p = -1, 0$ is then the sheaf property of $C^q(L,\mathbb{R},\mathcal{U})$, and vanishing for $p>0$ is tantamount to acyclicity of these sheaves.)

The second page of ${}_I E_r^{\bullet,\bullet}$ consists of the Čech cohomology of the Lie algebra cohomology presheaves $H^n(L,\mathbb{R})$, i.e., ${}_I E_2^{p,q} = \check{H}^p(H^q(L,\mathbb{R}),\mathcal{U})$. We obtain (recall that the row corresponding to $H^0(L,\mathbb{R})$ is not zero, but irrelevant)

0	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
0	$\check{H}^{-1}(H^3(L,\mathbb{R}),\mathcal{U})$	$\check{H}^{0}(H^3(L,\mathbb{R}),\mathcal{U})$	$\check{H}^{1}(H^3(L,\mathbb{R}),\mathcal{U})$	$\check{H}^{2}(H^3(L,\mathbb{R}),\mathcal{U})$	$\cdots$
0	$\check{H}^{-1}(H^2(L,\mathbb{R}),\mathcal{U})$	$\check{H}^{0}(H^2(L,\mathbb{R}),\mathcal{U})$	$\check{H}^{1}(H^2(L,\mathbb{R}),\mathcal{U})$	$\check{H}^{2}(H^2(L,\mathbb{R}),\mathcal{U})$	$\cdots$
0	$\check{H}^{-1}(H^1(L,\mathbb{R}),\mathcal{U})$	$\check{H}^{0}(H^1(L,\mathbb{R}),\mathcal{U})$	$\check{H}^{1}(H^1(L,\mathbb{R}),\mathcal{U})$	$\check{H}^{2}(H^1(L,\mathbb{R}),\mathcal{U})$	$\cdots$
0	0	0	0	0	

with a differential d_2 of bidegree $(2,-1)$, i.e. a mapping

$$d_2^{p,q} \,:\, \check{H}^p(H^q(L,\mathbb{R}),\mathcal{U}) \to \check{H}^{p+2}(H^{q-1}(L,\mathbb{R}),\mathcal{U})\,.$$

Since this spectral sequence too must converge to zero, we obtain

Proposition 1.15 *Let the precosheaf L be such that for each $i=1,\ldots,n$ the presheaves of Lie cochains with trivial coefficients $C^i(L,\mathbb{R})$ are in fact sheaves, and suppose that they satisfy $\check{H}^j(C^i(L,\mathbb{R}),\mathcal{U})=0$ for all $j\le 2(n-i)$. Suppose also that the cohomology $H^{n-1}(L,\mathbb{R})$ is a presheaf with $\check{H}^k(H^{n-k}(L,\mathbb{R}),\mathcal{U}) = \check{H}^{k+1}(H^{n-k}(L,\mathbb{R}),\mathcal{U})=0$ for $k=1,\ldots,n-1$. Then $H^n(L,\mathbb{R})$ is a sheaf.*

Proof : Under these conditions, the terms ${}_I E_r^{-1,n}$ and ${}_I E_r^{0,n}$ stabilise at $r=2$, and equal $\check{H}^{-1}(H^n(L,\mathbb{R}),\mathcal{U})$ and $\check{H}^0(H^n(L,\mathbb{R}),\mathcal{U})$ respectively. Indeed, the conditions have been so chosen that the maps d_r of degree $(r,1-r)$ always map to zero. Since ${}_I E_r^{p,q}$ must converge to zero, the result follows. □

For example, if $C^1(L,\mathbb{R})$ is a sheaf, then so is $H^1(L,\mathbb{R})$. Suppose that all $C^n(L,\mathbb{R})$ are acyclic sheaves. Suppose further that L is a sheaf of perfect Lie algebras (so that $H^1(L,\mathbb{R})=0$). Then $H^2(L,\mathbb{R})$ is a sheaf. This may help one determine $H^2(L,\mathbb{R})$ from local data. If one should find that $H^2(L,\mathbb{R})$ is an acyclic sheaf, then $H^3(L,\mathbb{R})$ must be a sheaf. Again, this information may help one determine it, and if it happens to be an acyclic sheaf, then $H^4(L,\mathbb{R})$ must be a sheaf as well, etc. etc.

2 Cosheaves of Lie algebras

This section is devoted to finding sufficient conditions in order that the presheaf of local continuous cochains be an acyclic sheaf. We first define cosheaves of Lie algebras.

Definition 2.1 *A precosheaf of Lie algebras is called a* cosheaf *if it further satisfies the (dual versions of) the local identity axiom and the gluing axiom.*

I If $\{U_i\}_{i\in I}$ is such that $\cup_I U_i = U$, then $L(U) = \sum_i \iota_{UU_i} L(U_i)$.

II If $\{U_i\}_{i\in I}$ is such that $\cup_I U_i = U$, and if $\sum_i \iota_{UU_i}(X_i) = 0$, then there exist $X_{ij} \in L(U_i \cap U_j)$ with $X_{ji} = -X_{ij}$ and $X_i = \sum_j \iota_{U_i U_{ij}}(X_{ij} - X_{ji})$.

A cosheaf is called flabby *if the ι_{UV} are all injective.*

The following property follows from II.

II' If $U = V_1 \cup V_2$, then $\iota_{UV_1}(L(V_1)) \cap \iota_{UV_2}(L(V_2)) = \iota_{UV_{12}}(L(V_{12}))$.

For a flabby precosheaf, II also follows from I and II'. A flabby cosheaf can therefore also be defined as a flabby precosheaf satisfying I and II'.

Proposition 2.2 *For a precosheaf of Lie algebras, II implies II'. For a flabby precosheaf, I and II' also imply II.*

Proof : We prove II', assuming the 'co-gluing' property II. If $Y \in \iota_{UV_1}(L(V_1)) \cap \iota_{UV_2}(L(V_2))$, then $Y = \iota_{UV_1}(X_1) = \iota_{UV_2}(-X_2)$ for some $X_1 \in L(V_1)$, $X_2 \in L(V_2)$. According to II, $\iota_{UV_1}(X_1) + \iota_{UV_2}(X_2) = 0$ then implies the existence of $X_{12} = -X_{21} \in L(U_{12})$ such that $X_1 = \iota_{U_1U_{12}}(X_{12} - X_{21})$ and $X_2 = \iota_{U_2U_{12}}(X_{21} - X_{12})$. Therefore $Y = \iota_{UU_{12}}(X_{12} - X_{21}) \in \iota_{UU_{12}}(L(U_{12}))$, and we have $\iota_{UV_1}(L(V_1)) \cap \iota_{UV_2}(L(V_2)) \subseteq \iota_{UV_{12}}(L(V_{12}))$. The converse inclusion is obvious.

Now we assume II', and prove II under the assumption that all the ι's are injective. We start with the case $N = 2$. If $\iota_{UU_1}(X_1) + \iota_{UU_2}(X_2) = 0$, then with $W = U_1 \cup U_2$ we have $\iota_{UW}(\iota_{WU_1}(X_1) + \iota_{WU_2}(X_2)) = 0$, and therefore $\iota_{WU_1}(X_1) + \iota_{WU_2}(X_2) = 0$ by injectivity of ι_{UW}. Thus $\iota_{WU_1}(X_1) = -\iota_{WU_2}(X_2)$ must be in $\iota_{WU_{12}}(L(U_{12}))$ by II' and we are done.

We proceed by induction on N. If $\iota_{UU_1}(X_1) + \ldots + \iota_{UU_N}(X_N) = 0$, then set $W := U_1 \cup \ldots \cup U_{N-1}$, and write

$$\iota_{UU_N}(X_N) = -\iota_{UW}(\iota_{WU_1}(X_1) + \ldots + \iota_{WU_{N-1}}(X_{N-1})) . \tag{3}$$

By property II', we have $\iota_{UU_N}(X_N) \in \iota_{UU_N} L(U_N) \cap \iota_{UW}(L(W)) = \iota_{UW\cap U_N}(L(W \cap U_N))$, and by property I, we have $L(W \cap U_N) = \sum_{i=1}^{N-1} \iota_{W\cap U_N U_{iN}}(L(U_{iN}))$. Setting $\iota_{UU_N}(X_N) = \iota_{UW\cap U_N}(Y_N)$ and $Y_N = \sum_{i=1}^{N-1} \iota_{W\cap U_N U_{iN}}(X_{iN} - X_{Ni})$ (with $X_{iN} = -X_{Ni}$), we therefore find

$$\iota_{UU_N}(X_N) = \sum_{i=1}^{N-1} \iota_{UU_{iN}}(X_{iN} - X_{Ni}) . \tag{4}$$

Decomposing $\iota_{UU_{iN}} = \iota_{UU_N} \iota_{U_N U_{iN}}$ and using the injectivity of ι_{UU_N}, we conclude

$$X_N = \sum_{i=1}^{N-1} \iota_{U_N U_{iN}}(X_{iN} - X_{Ni}) . \tag{5}$$

We can rewrite equation (3) as

$$0 = \sum_{i=1}^{N-1} \iota_{UU_i}\Big(X_i + \iota_{U_i U_{iN}}(X_{iN} - X_{Ni})\Big) .$$

We now apply the induction hypothesis, and obtain $X_{ij} = -X_{ji} \in L(U_{ij})$ (with $i, j \leq N-1$) such that

$$X_i + \iota_{U_i U_{iN}}(X_{iN} - X_{Ni}) = \sum_{j=1}^{N-1} \iota_{U_i U_{ij}}(X_{ij} - X_{ji})\,. \tag{6}$$

Combining equations (5) and (6), we obtain II. $\square$

We study cosheaves of Lie algebras over X with the following additional properties.

III Each $L(U)$ is perfect.

IV If $V \subseteq U$, then $\iota_{UV}(L(V))$ is an ideal in $L(U)$.

V $L(\emptyset) = \{0\}$.

In particular, if $V_1, V_2 \subseteq U$ and $V_1 \cap V_2 = \emptyset$, then

$$[\iota_{UV_1}(L(V_1)), \iota_{UV_2}(L(V_2))] = \iota_{U,V_1\cap V_2}(L(V_1 \cap V_2)) = \{0\}\,.$$

The only nontrivial commutators are the ones between 'overlapping' elements, so properties IV and V insure that the Lie bracket is local in nature. Property V excludes for example the flabby cosheaf $L(U) = \mathfrak{g} \ltimes C^\infty_c(U, \mathfrak{g})$, which has $L(\emptyset) = \mathfrak{g}$ as a 'global' component.

Note that for the natural precosheaf induced by a perfect Lie algebra L over $\mathrm{Spec}^m(L)$, II' and IV are automatically fulfilled. As it is flabby, I implies II. In this situation, it therefore suffices to check I, V, and III.

2.1 Localisation of cocycles revisited

The following should eventually be subsumed under theorem 1.12. It appears here separately because the proof is more or less in order, showing that the (important!) special case of second Lie algebra cohomology with trivial coefficients does not suffer from the holes in the 'proof' of theorem 1.12. Also, it shows how to handle the case where R is not $\mathbb{R}$.

The conditions I through V suffice to prove that the second Lie algebra cohomology with trivial coefficients is local.

Lemma 2.3 *Let L be a precosheaf of Lie algebras satisfying III, IV and V. Let R be a presheaf of representations of L which is local in the sense that $V \cap V' = \emptyset$ implies that $J_{V'U}\pi_U \circ \iota_{UV}$ is zero. Let ψ be an n-cocycle on $L(U)$ with values in $R(U)$. If $\xi_i \in \iota_{UU_i}(L(U_i))$ for $1 \leq i \leq n$ with $V \cap U_i = \emptyset$ for all i and $U_i \cap U_j = \emptyset$ for $i \neq j$, then*

$$J_{VU}\psi(\xi_1, \ldots, \xi_n) = 0\,.$$

In particular, if R is the trivial representation, then ψ lives on the fat diagonal. Under the above conditions, we have

$$\psi(\xi_1, \ldots, \xi_n) = 0\,.$$

Proof : As $L(U_1)$ is perfect, we may write $X_1 = \sum_m [Y_1^\alpha, Y_1^{'\alpha}]$ as a finite sum of commutators with $Y_1^\alpha, Y_1^{'\alpha} \in L(U_1)$. As $\delta\psi = 0$, we have

$$\sum_{0\leq k<l\leq n} (-1)^{k+l}\psi([\xi_k,\xi_l],\xi_0,\ldots,\hat{\xi}_k,\ldots,\hat{\xi}_l,\ldots,\xi_n) = -\sum_{k=0}^{n}(-1)^k\pi_U(\xi_k)\cdot\psi(\xi_0,\ldots,\hat{\xi}_k,\ldots,\xi_n)$$

for all $\xi_0, \ldots, \xi_n$ in $L(U)$. If we now substitute $\xi_0 = \iota_{UU_1}(Y_1^\alpha)$, $\xi_1 = \iota_{UU_1}(Y_1^{'\alpha})$, and $\xi_k = \iota_{UU_k}(X_k)$ for $k > 1$, then using the fact that $[\iota_{UU_i}(L(U_i)), \iota_{UU_j}(L(U_j))] = \{0\}$ for $1 \leq i < j \leq n$, we see that the only term surviving on the l.h.s. is $\psi([\iota_{UU_1}(Y_1^\alpha), \iota_{UU_1}(Y_1^{'\alpha})], \iota_{UU_2}(X_2), \ldots, \iota_{UU_n}(X_n))$. Thus

$$\begin{aligned}
&\psi(\iota_{UU_1}([Y_1^\alpha, Y_1^{'\alpha}]), \iota_{UU_2}(X_2), \ldots, \iota_{UU_n}(X_n)) = \\
&-\pi_U(\iota_{UU_1}(Y_1^\alpha))\psi(\iota_{UU_1}(Y_1^{'\alpha}), \iota_{UU_2}(X_2), \ldots, \iota_{UU_n}(X_n)) \\
&+\pi_U(\iota_{UU_1}(Y_1^{'\alpha}))\psi(\iota_{UU_1}(Y_1^\alpha), \iota_{UU_2}(X_2), \ldots, \iota_{UU_n}(X_n)) \\
&-\sum_{k=2}^{n}(-1)^k\pi_U(\iota_{UU_k}(X_k))\psi(\iota_{UU_1}(Y_1^\alpha), \iota_{UU_1}(Y_1^{'\alpha}), \iota_{UU_2}(X_2), \ldots, \hat{X}_k, \ldots, \iota_{UU_n}(X_n))\,.
\end{aligned}$$

As the r.h.s. is obviously contained in $\pi_U\big(\iota_{UU_1}(L(U_1)) + \ldots + \iota_{UU_n}(L(U_n))\big)R(U)$ for all α, so is $\psi(\iota_{UU_1}(X_1), \ldots, \iota_{UU_n}(X_n))$. Because $V \cap U_i = \emptyset$, the locality property of R insures that $J_{VU}\pi_U(\iota_{UU_i}(L(U_i)))$ is zero. □

Every 1-cochain with coefficients in $\mathbb{R}$ is local by definition. The above shows that closed 2-cochains with values in $\mathbb{R}$ are also local, so that $H^2(L,\mathbb{R}) = H^2_{\mathrm{loc}}(L,\mathbb{R})$ if L satisfies I through V.

2.2 Cohomology as a sheaf

2.2.1 Paracompact Hausdorff spaces

We gather some standard facts on paracompact Hausdorff spaces.

Proposition 2.4 *Every closed subset C of a paracompact Hausdorff space X is paracompact.*

Proof : Let $\mathcal{V}$ be an open cover of C. Then by definition, each $V \in \mathcal{V}$ is of the form $V' \cap C$ for some open $V' \subseteq X$. All these V', together with the open set $X - C$ constitute an open cover of X, which has a locally finite open refinement $\mathcal{W}$. Intersecting all $W \in \mathcal{W}$ with C yields a locally finite open refinement of $\mathcal{V}$. □

Proposition 2.5 *Every paracompact Hausdorff space X is regular; if $x \notin C$ for $C \subset X$ closed, then there exist open U and V with $x \in U$, $C \subset V$ and $U \cup V = \emptyset$. In particular, if $x \in U$ for an open U, then there exists an open neighbourhood V of x with $\overline{V} \subset U$.*

Proof : As X is Hausdorff, we may choose for each $c \in C$ disjoint open neighbourhoods U_c and V_c of x and c respectively. Let S_0 be the collection of all the V_c, supplemented by the set $X - C$. By paracompactness, S_0 has a locally finite refinement S_1. Let S_2 be the locally finite cover of C that one obtains by removing from S_1 the sets that do not intersect C. The point x then has a neighbourhood N intersecting only finitely many sets in S_2, say W_1 through

W_n. If W_i lies in V_{c_i}, let $U := N \cap \cap_{i=1}^n U_{c_i}$ and let V be the union of sets in S_2. Then surely $x \in U$, $C \subset V$ and $U \cup V = \emptyset$, as N only intersects the sets W_i, which have empty intersection with the U_{c_i}. □

If $\mathcal{V}$ is a covering of X, and $A \subset X$, then the *star* of A w.r.t. $\mathcal{V}$ is defined as $\mathrm{Star}(A, \mathcal{V}) := \bigcup\{V \in \mathcal{V}\,;\, V \cap A \neq \emptyset\}$. A refinement $\mathcal{V}$ of a covering $\mathcal{U}$ is called a *star-refinement* if for each $V \in \mathcal{V}$, there exists a $U \in \mathcal{U}$ such that $\mathrm{Star}(V, \mathcal{V}) \subseteq U$. The following theorem states that paracompactness can be defined in terms of star-refinements.

Theorem 2.6 *A Hausdorff space X is paracompact if and only if every open covering of X has an open star-refinement.*

Proof : See [Wil70, p. 151]. □

Definition 2.7 *A collection $\{U_1, \ldots, U_n\}$ of sets is called* connected *if for any $1 \leq i, j \leq n$, there exist $i = i_1, i_2, \ldots, i_{n-1}, i_k = j$ such that $U_{i_s} \cap U_{i_{s+1}} \neq \emptyset$ for all $1 \leq s \leq k - 1$.*

The following consequence is immediate, but nonetheless worth noting.

Corollary 2.8 *Let X be a paracompact Hausdorff space. Then for every open cover $\mathcal{U}$ of X, and for any $n \in \mathbb{N}$, there exists a locally finite cover $\mathcal{V}$ such that for every connected subcollection $\{V_1, \ldots, V_n\} \subset \mathcal{V}$, the union $\bigcup_{i=1}^n V_i$ is contained in some $U \in \mathcal{U}$.*

Proof : Iterate the procedure of getting a star-refinement of $\mathcal{U}$ $n - 1$ times, and take a locally finite refinement. □

2.2.2 Local cohomology

The following shows that the support of $X \in L(U)$ is always contained in a closed subset of U.

Proposition 2.9 *Let L be a precosheaf over a normal space X that satisfies property I. Let $U \subseteq X$ be open. Then for each $X \in L(U)$, there exists an open $U' \subset \overline{U'} \subset U$ with $X \in \iota_{UU'}(L(U'))$.*

Proof : The collection $\mathcal{V} = \{V \subset U\,;\, \overline{V} \subset U\}$ is a cover of U because X is regular. In view of property I, we can find $V_1, \ldots, V_N \in \mathcal{V}$ and $X_i \in L(V_i)$ such that $X = \sum_{i=1}^N \iota_{UV_i}(X_i)$. Thus $X = \iota_{UU'}(Y)$ with $U' = \bigcup_{i=1}^N V_i$ and $Y = \sum_{i=1}^N \iota_{U'V_i}(X_i)$. Clearly, we have $U \subset \overline{U'} \subset U$. □

Definition 2.10 *A cochain $\psi \in C^n(L(U), R(U))$ is called* local *if $\rho_{U_0U}\psi(X_1, \ldots, X_n) = 0$ for all $X_i \in \iota_{UU_i}(L(U_i))$, $i = 1, \ldots, n$, such that $\{U_0, U_1, \ldots, U_N\}$ is not connected. The vector space of local cochains is denoted $C^n_{\mathrm{loc}}(L(U), R(U))$.*

Note that for R the constant sheaf $R(U) = T$ with values in the trivial representation, this reduces to '$\psi(X_1, \ldots, X_n) = 0$ for all $X_i \in \iota_{UU_i}(L(U_i))$ such that $\{U_1, \ldots, U_N\}$ is not connected'. Also note that any collection containing $\emptyset$ is disconnected. Consequently, a local cochain ψ satisfies $\psi(X_1, \ldots, X_n) = 0$ as soon as *any* of the X_i is in $\iota_{U\emptyset}L(\emptyset)$. The above notion of 'locality' is a slightly weaker condition than being *diagonal* in the sense of Losev. Note that $U \mapsto C^n_{\mathrm{loc}}(L(U), R(U))$ is a sub-precosheaf of $U \mapsto C^n(L(U), R(U))$.

Lemma 2.11 *Let L be a cosheaf of Lie algebras over a paracompact Hausdorff space, and let R be a sheaf of representations. Then $U \mapsto C^n_{\mathrm{loc}}(L(U), R(U))$ is a sheaf for every $n \in \mathbb{N}$.*

Proof : The local identity axiom for $U \mapsto C^n_{\mathrm{loc}}(L(U), R(U))$ follows from the property I for the cosheaf L, and from local identity for R. Indeed, let ψ be a local n-cochain on $L(U)$, let $\mathcal{V}$ be an open cover of U such that $\iota^*_{UV}\psi = 0$ for all $V \in \mathcal{V}$. We prove that $\psi(X_1, \ldots, X_n) = 0$ for all $X_i \in L(U)$.

We would like to have star refinements of $\mathcal{V}$, but as U need not be paracompact, we will make due with star refinements of $\mathcal{V}' := \{V \cap U' \,;\, V \in \mathcal{V}\}$, with $U' \subset \overline{U'} \subset U$ a 'slightly smaller' set with the property that each of the X_i is of the form $X_i = \iota_{UU'}(Y_i)$ with $Y_i \in L(U')$ (cf. proposition 2.9). As $\{V \cap \overline{U'} \,;\, V \in \mathcal{V}\}$ is an open cover of the paracompact Hausdorff space $\overline{U'}$, it allows for a locally finite open $n+1$-fold star refinement $\mathcal{W}$ in the sense of corollary 2.8. Then $\mathcal{W}' := \{W \cap U' \,;\, W \in \mathcal{W}\}$ is a locally finite open $n+1$-fold star refinement of $\mathcal{V}'$.

Using property I, we write each Y_i as a finite sum $Y_i = \sum_{k_i=1}^{N_i} \iota_{U'W_{k_i}}(Y_i^{k_i})$, where each $Y_i^{k_i}$ is in $L(W_{k_i})$, and the W_{k_i} are in $\mathcal{W}'$. We then have

$$J_{W_{k_0}U}\psi(X_1, \ldots, X_n) = \sum_{k_1,\ldots,k_n} J_{W_{k_0}U}\psi\left(\iota_{UW_{k_1}}(Y_1^{k_1}), \ldots, \iota_{UW_{k_n}}(Y_n^{k_n})\right).$$

Since ψ is local, all terms on the right vanish except the ones where $\{W_{k_0}, W_{k_1}, \ldots, W_{k_n}\}$ is connected, in which case $W_{k_0} \cup \ldots \cup W_{k_n}$ is contained in a single $V \in \mathcal{V}$. Since $\iota^*_{UV}\psi = 0$, these terms must vanish too, and $J_{W_{k_0}U}\psi(X_1, \ldots, X_n) = 0$ for all W_{k_0}. Since the W_{k_0} cover U', the local identity axiom for R then tells us that $J_{U'U}\psi(X_1, \ldots, X_n) = 0$ for every U' with the property that all the X_i are in $\iota_{UU'}(L(U'))$. Now if we choose $U' \subseteq \overline{U}' \subseteq U'' \subseteq \overline{U}'' \subseteq U$, then $J_{U''U}\psi(X_1, \ldots, X_n) = 0$ because of the above, and $J_{(U-\overline{U}')U}\psi(X_1, \ldots, X_n) = 0$ because ψ is local, and $U - \overline{U}'$ is disjoint from the 'supports' of the X_i. Using again the local identity axiom for R, we see that $\psi(X_1, \ldots, X_n) = 0$ as required.

The gluing axiom for the $U \mapsto C^n_{\mathrm{loc}}(L(U), R(U))$ follows essentially from property II for the cosheaf L, and from the gluing axiom for R. Let $\mathcal{V}$ be a cover of U, and $\psi_V \in C^n_{\mathrm{loc}}(L(V), R(V))$ be such that $\iota^*_{VV\cap V'}\psi_V = \iota^*_{V'V\cap V'}\psi_{V'}$ for any $V, V' \in \mathcal{V}$. We wish to glue these together, i.e. we wish to find a (necessarily unique) $\psi_U \in C^n_{\mathrm{loc}}(L(U), T)$ such that $\iota^*_{UV}\psi_U = \psi_V$.

We fix $U' \subset \overline{U'} \subset U$, and first glue together the $\psi_{V'} := \iota^*_{VU'\cap V}\psi_V$ to obtain a $\psi_{U'}$ on $L(U')$. Again, let $\mathcal{W}'$ be an $n+1$-fold star refinement of $\mathcal{V}'$ (both covers of U'), and write $Y_i = \sum_{k_i=1}^{N_i} \iota_{U'W_{k_i}}(Y_i^{k_i})$, with $Y_i \in L(U')$ and $Y_i^{k_i}$ in $L(W_{k_i})$. We define $\psi_{k_0;k_1,\ldots,k_n}$ on $L(W_{k_1}) \times \ldots \times L(W_{k_n})$ to be zero if $\{W_{k_0}, W_{k_1}, \ldots, W_{k_n}\}$ is not connected. If it is connected, then $\bigcup_{i=0}^n W_{k_i} \subseteq V'$ for some $V' \in \mathcal{V}'$, and we define $\psi_{k_0;k_1,\ldots,k_n}$ to be the restriction of $J_{W_{k_0}V'}\psi_{V'}$. This does not depend on the choice of V'; if also $\bigcup_{i=0}^n W_{k_i} \subseteq V''$, then $\bigcup_{i=0}^n W_{k_i} \subseteq V' \cap V'$, and $\psi_{V'}$ agrees with $\psi_{V''}$ on $L(V' \cap V'')$ by assumption. We thus define

$$\psi_{k_0;U'}(Y_1, \ldots, Y_n) := \sum_{k_1,\ldots,k_n} \psi_{k_0;k_1,\ldots,k_n}(Y_1^{k_1}, \ldots, Y_n^{k_n}). \tag{7}$$

We need to check that this is independent of the way we split Y_i into $\iota^*_{U'W_{k_i}}(Y_i^{k_i})$. Suppose that also $Y_i = \sum_{l_i} \iota^*_{U'W_{l_i}}({Y'}_i^{l_i})$. Without loss of generality, we can assume that the labels l_i and k_i are the same. Then the difference between the two versions of equation (7) is

$$\sum_{k_1,\dots,k_n} \begin{aligned}[t] & \psi_{k_0;k_1,\dots,k_n}(Y_1^{k_1} - Y'^{k_1}_1, Y_2^{k_2}, \dots, Y_n^{k_n}) \\ + \; & \psi_{k_0;k_1,\dots,k_n}(Y'^{k_1}_1, Y_2^{k_2} - Y'^{k_2}_2, Y_3^{k_3}, \dots, Y_n^{k_n}) \\ + \; & \dots \\ + \; & \psi_{k_0;k_1,\dots,k_n}(Y'^{k_1}_1, \dots, Y'^{k_{n-1}}_{n-1}, Y_n^{k_n} - Y'^{k_n}_n)\,. \end{aligned}$$

Let $Z_i^k := Y_i^k - Y'^k_i$. Then $\sum_{k_i} \iota_{U'W_{k_i}}(Z_i^{k_i}) = 0$, so that the property II for L yields $Z_i^{kl} \in L(W_k \cap W_l)$ such that $Z_i^{kl} = -Z_i^{lk}$ and $Z_i^k = \sum_l \iota_{W_k W_k \cap W_l}(Z_i^{kl})$. Consequently, each nonzero term

$$\psi_{k_0;k_1,\dots,k_n}(Y'^{k_1}_1, \dots, \iota_{W_{k_i} W_{k_i} \cap W_{l_i}}(Z_i^{k_i l_i}) \dots, Y_n^{k_n})$$

coming from $\psi_{k_0;k_1,\dots,k_n}(Y'^{k_1}_1, \dots, Z_i^{k_i} \dots, Y_n^{k_n})$ is compensated by a term

$$\psi_{k_0;k_1,\dots,l_i,\dots,k_n}(Y'^{k_1}_1, \dots, \iota_{W_{l_i} W_{k_i} \cap W_{l_i}}(-Z_i^{k_i l_i}), \dots, Y_n^{k_n})$$

coming from $\psi_{k_0;k_1,\dots,k_n}(Y'^{k_1}_1, \dots, Z_i^{l_i} \dots, Y_n^{k_n})$. Indeed, if the former is nonzero, then the collection $\{W_{k_0}, \dots, W_{k_i} \cap W_{l_i}, \dots W_{k_n}\}$ is connected (and in particular $W_{k_i} \cap W_{l_i} \neq \emptyset$), so that $\{W_{k_0}, \dots, W_{k_n}\} \cup \{W_{l_i}\}$ is connected. Therefore, $W_{l_i} \cup (W_{k_0} \cup \dots \cup W_{k_n})$ is contained in a single set $V' \in \mathcal{V}'$ (remember that $\mathcal{V}'$ was an $n+1$-fold star refinement rather than an n-fold), and $\psi_{k_0;k_1,\dots,k_n}$ agrees with $\psi_{k_0;k_1,\dots,l_i,\dots,k_n}$. Every nonzero term in the difference is thus cancelled by another term, and $\psi_{k_0;U'}(Y_1, \dots, Y_n)$ is a well defined element of $R(W_{k_0})$. If $W_{k_0} \cap W_{k'_0} \neq \emptyset$, then $W_{k_0} \cup W_{k'_0} \subseteq V'$ for some $V' \in \mathcal{V}$, so that $\psi_{k_0;U'}(Y_1, \dots, Y_n)$ and $\psi_{k'_0;U'}(Y_1, \dots, Y_n)$ agree on $W_{k_0} \cap W_{k'_0}$. We then use the gluing axiom on R to assemble the $\psi_{k_0;U'}(Y_1, \dots, Y_n)$ into a single well defined $\psi_{U'}(Y_1, \dots, Y_n)$.

It is clear from the definition that $\iota^*_{U'V'}\psi_{U'} = \psi_{V'}$. Indeed, let all the Y_i be in $\iota_{U'V'}L(V')$. Then the $W \cap V'$ with $W \in \mathcal{W}'$ cover V', and the $Y_i^{k_i}$ can be chosen as $\iota_{W_{k_i} W_{k_i} \cap V'} Y'^{k_i}_i$ with $Y'^{k_i}_i \in L(W_{k_i} \cap V')$. Assume that $\{W_{k_0}, \dots, W_{k_n}\}$ is connected. If $\bigcup_{i=0}^n W_{k_i} \subseteq V''$, then $\bigcup_{i=0}^n (W_{k_i} \cap V') \subseteq V'' \cap V'$, so that $J_{V' \cap W_{k_0} V''}\psi_{V''}(\iota_{V''W_{k_1}} Y_1^{k_1}, \dots, \iota_{V''W_{k_n}} Y_n^{k_n})$ is equal to $J_{V' \cap W_{k_0} V'}\psi_{V'}(\iota_{V'W_{k_1} \cap V'}(Y'^{k_1}_1), \dots, \iota_{V'W_{k_n} \cap V'}(Y'^{k_n}_n))$ due to the requirement that $\psi_{V'}$ and $\psi_{V''}$ agree on the overlap of V' and V''. This means that if the Y_i come from V', then all the $\psi_{k_0;k_1,\dots,k_n}(Y_1^{k_1}, \dots, Y_n^{k_n})$ can be expressed in terms of $\psi_{V'}$, so that $\iota^*_{U'V'}\psi_{U'} = \psi_{V'}$.

We have shown that the $\psi_{V'}$ glue together to a $\psi_{U'}$ on $L(U')$. In order to extend this to U, we need property I. Because of 'local identity', the $\psi_{U'}$ is unique, and does not depend on our choice of refinement. If X_i is in $L(U)$ for $i = 1, \dots, n$, we choose $U' \subset \overline{U'} \subset U$ such that $X_i = \iota_{UU'}(Y_i)$, and set $\psi(X_1, \dots, X_n) := \psi_{U'}(Y_1, \dots, Y_n)$. This does not depend on our choice of U'; if U'' is another possibility, then $U''' = U' \cap U''$ is yet another, and $\iota^*_{U'U'''}\psi_{U'} = \iota^*_{U''U'''}\psi_{U''} = \psi_{U'''}$, because of the uniqueness of $\psi_{U'''}$. □

Proposition 2.12 *Let L be a cosheaf of Lie algebras over a paracompact Hausdorff space satisfying IV. Let R be a sheaf of representations with the property that $V \cap V' = \emptyset$ implies $J_{V'U}\pi_U(\iota_{UV}(X_V)) = 0$. Then $\delta_U : C^n_{\mathrm{loc}}(L(U), R(U)) \to C^{n+1}_{\mathrm{loc}}(L(U), R(U))$ is a homomorphism of sheaves.*

Proof : We already know that δ_U is a morphism of presheaves $C^n(L(U), R(U)) \to C^{n+1}(L(U), R(U))$, and we need only show that the image of a local cocycle is local. Consider $J_{U_{-1}U}\delta\psi(X_0, \ldots, X_n)$ with $X_i \in \iota_{UU_i}(L(U_i))$, and suppose that $\{U_{-1}, \ldots, U_n\}$ is not connected.

Consider first the terms of the form $J_{U_{-1}U}\psi([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_n)$. Because of property IV, $[\iota_{U_i \cup U_j U_i}(L(U_i)), \iota_{U_i \cup U_j U_j}(L(U_j))]$ is contained in $\iota_{U_i \cup U_j U_i}(L(U_i)) \cap \iota_{U_i \cup U_j U_j}(L(U_j))$. Because of property II, $\iota_{U_i \cup U_j U_i}(L(U_i)) \cap \iota_{U_i \cup U_j U_j}(L(U_j))$ is contained in $\iota_{U_i \cup U_j U_i \cap U_j}(L(U_i \cap U_j))$. But $\{U_i \cap U_j, U_{-1}, \ldots, \hat{U}_i, \ldots, \hat{U}_j, \ldots, U_n\}$ cannot be connected: if $U_i \cap U_j = \emptyset$, then this is clear. if $U_i \cap U_j \neq \emptyset$, then this follows from the fact that $\{U_{-1}, \ldots, U_n\}$ was not connected. So either way, the terms of the form $J_{U_{-1}U}\psi([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_n)$ vanish.

Now suppose that any term of the form $J_{U_{-1}U}\pi_U(\iota_{UU_i}(X_i))\psi(X_0, \ldots, \hat{X}_i, \ldots, X_n)$ is nonzero. Then $\{U_0, \ldots, \hat{U}_i, \ldots, U_n\}$ is connected. Also, $\{U_{-1}, U_i\}$ must be connected because of the 'locality' condition we imposed on R. And finally, $\{U_{-1} \cup U_i, U_0, \ldots, \hat{U}_i, \ldots, U_n\}$ is connected because $J_{U_{-1}\cup U_i U}\pi_U(\iota_{UU_{-1}\cup U_i}(\iota_{U_{-1}\cup U_i U_i}X_i))\psi(X_0, \ldots, \hat{X}_i, \ldots, X_n)$ is nonzero, and equal to the expression $\pi_{U_{-1}\cup U_i}(\iota_{U_{-1}\cup U_i U_i}X_i)J_{U_{-1}\cup U_i U}\psi(X_0, \ldots, \hat{X}_i, \ldots, X_n)$. All of this entails that $\{U_{-1}, \ldots, U_n\}$ is connected, contrary to our assumption. □

This is in general not sufficient to prove that the local cohomology $H^n_{\text{loc}}(L(U), R(U))$ is a sheaf. The following appears to be a convenient way to guarantee that the chains are acyclic.

Proposition 2.13 *Let X be a paracompact Hausdorff space, let L be a flabby cosheaf of Lie algebras satisfying IV and V, and and let R be a sheaf of representations. Suppose that L has* partitions of unity*, i.e. that for every cover $\{U_i\}$ of U, there exist linear maps $\sigma_i : L(U) \to L(U_i)$ such that for every $X \in L(U)$, only finitely many $\sigma_i(X)$ are nonzero, and $X = \sum_i \iota_{UU_i}\sigma_i(X)$. Suppose furthermore that these partitions of unity are* local *in the sense that $\sigma_{U_i} \circ \iota_{UU_j} = 0$ if $U_i \cap U_j = \emptyset$. Then the sheaves $U \mapsto C^n_{\text{loc}}(L(U), R(U))$ are soft, and therefore acyclic.*

Remark 2.14 *One could of course try to use Hahn-Banach in order to prove that the chains constitute even a* flabby *sheaf. Every time you use the axiom of choice though, a little kitten dies and goes to heaven. (I refuse to specify which one.)*

Proof : We wish to prove that the restriction of $C^n_{\text{loc}}(L(U), R(U))$ to a *closed* set $G \subset U$ is surjective. A section of the sheaf of cochains over G is precisely an element of $\varinjlim_{V \supset G} C^n_{\text{loc}}(L(V), R(V))$. So choose $V \supset G$, and take an element $\psi_V \in C^n_{\text{loc}}(L(V), R(V))$ that represents the germ. Choose $G \subset V' \subset \overline{V}' \subset V$ (X is normal, so you can do this). Then $U - \overline{V}'$, and V cover U, so choose $\sigma_{U-\overline{V}'}$ and σ_V such that $\sigma_{U-\overline{V}'}(X) + \sigma_V(X) = 0$ for all $X \in L(U)$. If $X \in L(V')$, then $\sigma_{U-\overline{V}'}(X) = 0$, so $X = \sigma_V(X)$. The cochain $\psi_U := \sigma_V^*\psi_V$ is thus an extension of the germ of ψ_V over G. □

We formulate proposition 1.15 for the local cohomology.

Proposition 2.15 *Let the precosheaf L be such that the sheaves $U \mapsto C^n_{\text{loc}}(L(U), \mathbb{R})$ are acyclic. (e.g., L may satisfy the hypotheses of proposition 2.13.) Suppose also that the local cohomology $H^{n-1}_{\text{loc}}(L, \mathbb{R})$ is a sheaf with $\check{H}^k(H^{n-1}_{\text{loc}}(L, \mathbb{R}), \mathcal{U}) = 0$ for $k = 1, 2$ and for all coverings $\mathcal{U}$. Then H^n_{loc} is a sheaf.*

Proof : Repeat the reasoning leading up to proposition 1.15, replacing the Lie cochains $C^k(L,\mathbb{R})$ by local cochains $C_{\mathrm{loc}}(L,\mathbb{R})$. $\square$

2.3 Synthesis

Let $\mathcal{F}$ be a presheaf. After a choice of cover $\mathcal{V} = \{V_i\}_{i\in I}$ such that $\bigcup_{i\in I} V_i = U$, we denote by $\check{H}^{-1}\mathcal{F}(\mathcal{V})$ and $\check{H}^0\mathcal{F}(\mathcal{V})$ the cohomologies of the sequence

$$0 \to \mathcal{F}(U) \to \check{C}^0\mathcal{F}(\mathcal{V}) \to \check{C}^1\mathcal{F}(\mathcal{V})\,. \tag{8}$$

Of course $\mathcal{F}$ satisfies the 'local identity' axiom if and only if $\check{H}^{-1}\mathcal{F}(\mathcal{V})$ vanishes for all possible covers, and the 'gluing' axiom if $\check{H}^0\mathcal{F}(\mathcal{V})$ does. In effect, $\check{H}^{-1}\mathcal{F}(\mathcal{V})$ and $\check{H}^0\mathcal{F}(\mathcal{V})$ measure how far $\mathcal{F}$ is removed from being a sheaf.

The following (well known) proposition says that two presheaves are isomorphic if they are isomorphic locally, and if they are equally far removed from being a sheaf.

Proposition 2.16 *Let $\mathcal{F}$ and $\mathcal{S}$ be presheaves over X, let $\mathcal{V} = \{V_i\}_{i\in I}$ be an open cover of $U \subseteq X$, and let $\mu : \mathcal{F} \to \mathcal{S}$ be a morphism of presheaves such that*

- *μ is a local isomorphism, i.e. $\mu_{V_i} : \mathcal{F}(V_i) \to \mathcal{S}(V_i)$ is an isomorphism for all $i \in I$.*
- *The induced map $\check{H}^i\mu : \check{H}^i\mathcal{F}(\mathcal{V}) \to \check{H}^i\mathcal{S}(\mathcal{V})$ is an isomorphism for $i = -1$, and is injective for $i = 0$.*

Then μ an isomorphism of presheaves.

Proof : We show that $\mu_U : \mathcal{F}(U) \to \mathcal{S}(U)$ is an isomorphism.

First, we show that μ_U is injective. Suppose that $\mu_U(f_U) = 0$ in $\mathcal{S}(U)$. Then certainly $\rho_{V_iU}\mu_U(f_U) = \mu_{V_i}\rho_{V_iU}(f_U) = 0$ for all $i \in I$, and since μ_{V_i} is an isomorphism, we have $f_{V_i} := \rho_{V_iU}(f_U) = 0$. Thus f_U defines a class in $\check{H}^{-1}\mathcal{F}(\mathcal{V})$, and since $\check{H}^{-1}\mu$ is injective, $\check{H}^{-1}\mu([f_U]) = [\mu_U(f_U)] = 0$ implies $[f_U] = 0$ in $\check{H}^{-1}\mathcal{F}(\mathcal{V})$. But then $f_U = 0$, and thus μ_U is injective.

Next, we show that μ_U is surjective. Given $s_U \in \mathcal{S}(U)$, we construct an $f_U \in \mathcal{F}(U)$ such that $\mu_U(f_U) = s_U$. Set $s_i := \rho_{V_iU}(s_U)$, so $\rho_{V_{ij}V_i}(s_i) = \rho_{V_{ij}V_j}(s_j)$ by the presheaf property of $\mathcal{S}$. (We write $V_{ij} = V_i \cap V_j$.) Set $f_i := \mu^{-1}_{V_i}(s_i)$ and observe $\mu_{V_{ij}}\rho_{V_{ij}V_i}(f_i) = \rho_{V_{ij}V_i}(s_i) = \rho_{V_{ij}V_j}(s_j) = \mu_{V_{ij}}\rho_{V_{ij}V_j}(f_j)$. Since $\mu_{V_{ij}}$ is an isomorphism, this implies $\rho_{V_{ij}V_i}f_i = \rho_{V_{ij}V_j}f_j$. The f_i constitute a Čech -cocycle in $\check{C}^0\mathcal{F}(\mathcal{V})$, and $[f_i]$ is a class in $\check{H}^0\mathcal{F}(\mathcal{V})$. Since $\check{H}^0\mu([f_i]) = [s_i] = 0$ and $\check{H}^0\mu$ is injective, we have $[f_i] = 0$. So there exists an $f'_U \in \mathcal{F}(U)$ with $\rho_{V_iU}f'_U = f_i$. Thus $\rho_{V_iU}(\mu_U(f'_U) - s_U) = 0$, and $[\mu_U(f'_U) - s_U] \in \check{H}^{-1}\mathcal{S}(\mathcal{V})$. Since $\check{H}^{-1}\mu$ is surjective, we can pick $[f''_U]$ such that $\check{H}^{-1}\mu([f''_U]) = [\mu_U(f'_U) - s_U]$, so that $[\mu_U(f'_U - f''_U) - s_U] = 0$. Thus with $f_U = f'_U - f''_U$, we have $\mu_U(f_U) = s_U$, and μ_U is surjective. $\square$

If $\mathcal{F}$ is a sheaf and $\mathcal{S}$ is a monopresheaf, we have $\check{H}^{-1}\mathcal{F}(\mathcal{V}) = \check{H}^{-1}\mathcal{S}(\mathcal{V}) = 0$ and $\check{H}^0\mathcal{F}(\mathcal{V}) = 0$, so that the second requirement is automatically satisfied. We obtain the following well known corollary.

Corollary 2.17 *Let $\mathcal{F}$ be a sheaf, $\mathcal{S}$ a monopresheaf (i.e. a presheaf that satisfies the local identity axiom), and let $\mu : \mathcal{F} \to \mathcal{S}$ be a morphism of presheaves such that each $x \in M$ has an open neighbourhood V such that $\mu_W : \mathcal{F}(W) \to \mathcal{S}(W)$ is an isomorphism for any open $W \subseteq V$. Then $\mathcal{S}$ is a sheaf, and μ an isomorphism.*

3 Examples

Let (X, ω) be a symplectic manifold of dimension $2n$. We introduce 4 subtly different Lie algebras of compactly supported infinitesimal symmetries of (X, ω). The symplectic Lie algebra is defined as

$$\mathrm{Sp}_c(X) := \{X \in \mathrm{Vec}_c(X)\,;\, L_X\omega = 0\}\,.$$

In particular, since $d\omega = 0$, $L_X\omega = di_X\omega = 0$. If $i_X\omega$ is not only closed but also exact, $i_X\omega = -df$, then X is called Hamiltonian:

$$\mathrm{Ham}_c(X) = \{X \in \mathrm{Vec}_c(X)\,;\, \exists f \in C^\infty(X) \text{ s.t. } df = -i_X\omega\}\,.$$

We define $C^\infty_c \to \mathrm{Ham}_c(X)$ by mapping f to the unique X_f such that $df = -i_{X_f}\omega$. Note that f in the definition of $\mathrm{Ham}_c(X)$ need not be compactly supported, so that $C^\infty_c \to \mathrm{Ham}_c(X)$ need not be surjective if X is noncompact. We equip C^∞_c with the Poisson bracket $\{f, g\} = \omega(X_f, X_g) = X_f(g)$, so that $f \mapsto X_f$ becomes a homomorphism. Finally, we define

$$N(X) := \{f \in C^\infty_c(X)\,;\, \exists \psi \in \Omega^{2n-1}_c(X) \text{ s.t. } f\omega^{\wedge \mathrm{n}} = d\psi\}\,.$$

The relations between $N(X)$, $C^\infty_c(X)$ and $\mathrm{Sp_c(X)}$ are neatly summarised by the exact sequences

$$0 \to H^0_c(X, \mathbb{R}) \to C^\infty_c(X) \to \mathrm{Sp}_c(X) \to H^1_c(X, \mathbb{R}) \to 0\,, \qquad (9)$$

$$0 \to N(X) \to C^\infty_c(X) \to H^{2n}_c(X, \mathbb{R}) \to 0\,, \qquad (10)$$

and

$$0 \to N(X) \to \mathrm{Sp}_c(X) \to H^1_c(X, \mathbb{R}) \oplus H^{2n}_c(X, \mathbb{R})/H^0_c(X, \mathbb{R}) \to 0\,. \qquad (11)$$

The third equation is obtained from the first two by noting that if $df = 0$, then $f\omega^{\wedge \mathrm{n}}$ restricted to each connected component X_i must be a multiple of ω^n. Since $[\omega^n] \neq 0$ in $H^{2n}(X_i, \mathbb{R})$, the volume form $f\omega^{\wedge \mathrm{n}}$ cannot be exact unless it's zero. Thus $N(X) \to \mathrm{Sp}_c(X)$ is injective. Quotienting (9) and (10) by $N(X)$, we obtain

$$0 \to H^{2n}_c(X, \mathbb{R})/H^0_c(X, \mathbb{R}) \to \mathrm{Sp}_c(X)/N(X) \to H^1_c(X, \mathbb{R}) \to 0$$

Since the terms on the right and those on the left have commuting representatives, equation (11) follows.

Note that $N(X)$ is isomorphic to the image of $C^\infty_c(X)$ in $\mathrm{Ham}_c(X)$ if X is compact, so in that case $N(X) \simeq \mathrm{Ham}(X)$.

Proposition 3.1 *The commutator ideal in* $\mathrm{Ham}_c(X)$, $C^\infty_c(X)$ *or* $N(X)$ *is precisely the image of* $N(X)$*. In particular, the Lie algebra* $N(X)$ *is perfect,* $[N(X), N(X)] = N(X)$*.*

Proof : Suppose $f \in C^\infty(X)$ such that $f = \{g, h\}$. Then because $L_{X_g}\omega = 0$, we have

$$\begin{aligned} f \cdot \omega^{\wedge \mathrm{n}} &= X_g(h) \cdot \omega^{\wedge \mathrm{n}} \\ &= L_{X_g}(h \cdot \omega^{\wedge \mathrm{n}}) \\ &= d(h \cdot i_{X_g}\omega^{\wedge \mathrm{n}}) \\ &= -n\, d(h \cdot dg \wedge \omega^{\wedge(n-1)}) \\ &= -n\, dh \wedge dg \wedge \omega^{\wedge(n-1)}. \end{aligned}$$

In particular, $f\omega^{\wedge n}$ is exact ($f\omega^{\wedge n} = d\psi$) if $f = \sum_{i=1}^{n}\{f_i, g_i\}$ with $f_i, g_i \in C^{\infty}(X)$. If furthermore X_{g_i} and X_{h_i} are in $\mathrm{Ham}_c(X)$, i.e. if df_i and dg_i are compactly supported, then clearly ψ can be chosen to be compactly supported as well.

Conversely, suppose that $f\omega^{\wedge n} = d\psi$ with ψ compacty supported. We show that X_f is in the commutator ideal. Write $\psi = \sum_{k=1}^{m}\psi_k$, where ψ_k has compact support in an area with Darboux coordinates x^i, p^i. Note that $dx^i \wedge \omega^{\wedge(n-1)}$ and $dp^i \wedge \omega^{\wedge(n-1)}$ constitute a basis for $\wedge^{2n-1}TX_m$ at each point, so that we can write $\psi_k = \sum_{i=1}^{n}\phi_k^i dx^i \wedge \omega^{\wedge(n-1)} + \chi_k^i dp^i \wedge \omega^{\wedge(n-1)}$, with ϕ_k^i and χ_k^i compactly supported. Then choose compactly supported ξ_k^i and η_k^i that equal x^i and p^i on the support of ϕ_k^i and χ_k^i respectively to obtain $\psi_k = \sum_{i=1}^{n}\phi_k^i d\xi_k^i \wedge \omega^{\wedge(n-1)} + \chi_k^i d\eta_k^i \wedge \omega^{\wedge(n-1)}$, and thus $f = -\frac{1}{n}\sum_{i=1}^{n}\sum_{k=1}^{m}\{\phi_k^i, \xi_k^i\} + \{\chi_k^i, \eta_k^i\}$. □

3.0.1 The Hamiltonian functions

Because of equation (10) and because the commutator ideal of $C_c^{\infty}(X)$ is $N(X)$, we have $H^1_{LA}(C_c^{\infty}(X), \mathbb{R}) \simeq H_c^{2n}(X, \mathbb{R})^*$. Note that $U \mapsto C_c^{\infty}(U)$ is a cosheaf with partitions of unity, so that the sheaves of Lie cochains are acyclic. Consequently, $H^1_{LA}(C_c^{\infty}(X), \mathbb{R})$ is a sheaf. (This can be checked independently.)

We assume that a cover $\{U_i\}$ of X has been chosen such that all intersections are either empty or star-shaped, so that $H_c^{2n}(U_{i_1,\ldots,i_n}, \mathbb{R}) \simeq \mathbb{R}$. We then have $\check{H}^n(X, H^1_{LA}(C_c^{\infty}, \mathbb{R})) \simeq \check{H}^n(X, \underline{\mathbb{R}})$. (For $n \geq 0$, and if X is connected also for $n = -1$.) In view of the spectral sequence described before, the kernel and cokernel of $d_2^{(-1,2)} : \check{H}^{-1}H^2_{LA} \to \check{H}^1 H^1_{LA}$ are the $(-1, 2)$ and $(1, 1)$ terms on the third page, and thus survive to infinity. Since $E_n^{p,q}$ converges against zero, they both vanish, so that $\check{H}^{-1}H^2_{LA} \simeq \check{H}^1(X, \underline{\mathbb{R}})$. Similarly, the kernel of $d_2^{(0,2)} : \check{H}^0 H^2_{LA} \to \check{H}^2 H^1_{LA}$ survives, so that $\check{H}^0 H^2_{LA} \hookrightarrow H^2(X, \mathbb{R})$ is injective.

Remark 3.2 *The third page shows that the cokernel of* $d_2^{(0,2)}$ *is isomorphic to the kernel of* $d_2^{-1,3}$ *in* $\check{H}^{-1}H^3_{LA}$. *If, as suspected, we have* $\check{H}^n H^2_{LA} = 0$ *for* $n \geq 0$, *then the third and fourth page of the spectral sequence show that* $\check{H}^{-1}H^3_{LA} \simeq \check{H}^2(X, \underline{\mathbb{R}})$ *and that* $d_3^{0,3} : \check{H}^0 H^3_{LA} \to \check{H}^3(X, \underline{\mathbb{R}})$ *is injective.*

3.0.2 The algebra $N(X)$

Since $N(X)$ is perfect, we have $H^1(N(X), \mathbb{R}) = \{0\}$. Because (10) is an exact sequence, and C_c^{∞} and H_c^{2n} are cosheaves, we have that N is an epiprecosheaf.

Proposition 3.3 *The assignment* $U \mapsto H_c^{2n}(U, \mathbb{R})$ *is a cosheaf.*

Proof : If U_i covers U, then every $f_U\omega^{\wedge n}$ can be written as $\sum_i f_i\omega^{\wedge n}$ using a partition of unity. We prove that if $\iota_{U\cup V,U}([f_U\omega^{\wedge n}]) = \iota_{U\cup V,V}(f_V\omega^{\wedge n})$, then $[f_U\omega^{\wedge n}]$ and $[f_V\omega^{\wedge n}]$ have representatives with support in $U \cup V$. If $f_U\omega^{\wedge n} - f_V\omega^{\wedge n} = d\psi_{U\cup V}$, write $\psi_{U\cup V} = \psi_U - \psi_V$ using partitions of unity. Then $f_U\omega^{\wedge n} - d\psi_U = f_V\omega^{\wedge n} - d\psi_V$ has support in $U \cap V$. □

Consequently, $C^1(N, \mathbb{R})$ is a monopresheaf, $\check{H}^{-1}C^1(N, \mathbb{R}) = 0$. The exact sequence of presheaves

$$0 \to C^1(H_c^{2n}, \mathbb{R}) \to C^1(C_c^{\infty}, \mathbb{R}) \to C^1(N, \mathbb{R}) \to 0$$

with the middle one acyclic yields an isomorphism

$$\check{H}^k(C^1(N),\mathbb{R}) \simeq \check{H}^{k+1}((H_c^{2n})^*)$$

One can check that indeed $\check{H}^0((H_c^{2n})^*) = \{0\}$, in agreement with $\check{H}^{-1}C^1(N,\mathbb{R}) = 0$. If $k \neq -1$, then one can take $(H_c^{2n}(U_{i_1,\dots,i_n}))^*) \simeq \mathbb{R}$, and

$$\check{H}^k(C^1(N),\mathbb{R}) \simeq H^{k+1}(X,\mathbb{R})\,.$$

3.1 A morphism of sheaves

If $\mathfrak{g}$ is any Lie algebra with an invariant bilinear symmetric form κ, i.e. $\kappa([X,Y],Z)+\kappa(Y,[X,Z]) = 0$, then every antisymmetric (w.r.t. κ) derivation S of $\mathfrak{g}$ induces a 2-cocycle ψ_S by $\psi_S(X,Y) := \kappa(S(X),Y)$. Indeed, since

$$\begin{aligned}
\delta\psi_S(X,Y,Z) &= \kappa(S([X,Y]),Z) + \text{cycl.}\\
&= \kappa([S(X),Y],Z) + \kappa([X,S(Y)],Z) + \text{cycl.}\\
&= \kappa([Y,Z],S(X)) + \kappa([Z,X],S(Y)) + \text{cycl.}\\
&= 2\kappa([X,Y],S(Z)) + \text{cycl}\\
&= -2\kappa(S([X,Y]),Z) + \text{cycl}\\
&= -2\delta\psi_S(X,Y,Z)\,,
\end{aligned}$$

we must have $\delta\psi_S = 0$. If S happens to be an inner derivation, $S = [Z,\bullet]$, then $\psi_S(X,Y) = \kappa([Z,X],Y) = \kappa(Z,[X,Y]) = \delta\chi_Z(X,Y)$ with $\chi_Z(X) := \kappa(Z,X)$. We thus have a map $\mathrm{Out}(\mathfrak{g})_{AS} \to H^2(\mathfrak{g},\mathbb{R})$ from the antisymmetric outer derivations of $\mathfrak{g}$ into the second cohomology.

3.1.1 Antisymmetric derivations for $N(X)$ and $C_c^\infty(X)$

We have seen that $C_c^\infty(X) \simeq N(X) \oplus \mathfrak{z}$ with centre $\mathfrak{z} \simeq H_c^{2n}(X,\mathbb{R})$ consisting of the compactly supported functions which are constant on every connected component (and thus zero on every noncompact component). Since $N(X)$ is perfect, we have

$$H^2(C_c^\infty(X),\mathbb{R}) = H^2(N(X),\mathbb{R}) \oplus \wedge^2 H_c^{2n}(X,\mathbb{R})^*\,.$$

Since $N(X)$ is perfect, its second Lie algebra cohomology is local, and the above direct sum embodies a splitting of the cohomology in a local part and a part generated by the cohomology in degree 1. In particular, the second (nonlocal) term dies for connected X, compact or not.

For $\mathfrak{g} = N(X)$ or $\mathfrak{g} = C_c^\infty(X)$, we have the *nondegenerate* invariant bilinear form

$$\kappa(f,g) = \int_X fg\,\omega^{\wedge \mathrm{n}}\,.$$

Bilinearity is clear, and it's invariant because $(\{h,f\}g+f\{h,g\})\omega^{\wedge \mathrm{n}} = \{h,fg\}\omega^{\wedge \mathrm{n}} = (L_{X_h}(fg))\omega^{\wedge \mathrm{n}} = L_{X_h}(fg\omega^{\wedge \mathrm{n}}) = d(fgi_{X_h}\omega^{\wedge \mathrm{n}})$. Nondegeneracy follows because $f^2\omega^{\wedge \mathrm{n}}$ is a *positive* volume form.

Every symplectic vector field $S \in \mathrm{Sp}(X)$ (compactly supported or not!) induces an antisymmetric derivation $f \mapsto L_S f$ on $C_c^\infty(X)$ that restricts to $N(X)$. We first check that S maps $N(X)$ to $N(X)$. If $f \in N(X)$, i.e. $f\omega^{\wedge \mathrm{n}} = d\psi$, with

ψ compactly supported, then $(L_S f)\omega^{\wedge n} = L_S(f\omega^{\wedge n}) = L_S d\psi = d(i_S d\psi)$ and $L_S f$ is again an element of $N(X)$. We check that S is a derivation on $C^\infty_c(X)$. If X_f is a Hamilton vector field, then $L_S f$ is a Hamilton function for $[S, X_f]$. Indeed, $dL_S f = di_S df = -di_S i_{X_f}\omega = di_{X_f} i_S\omega = L_{X_f} i_S\omega = i_S L_{X_f}\omega + i_{[X_f,S]}\omega = -i_{[S,X_f]}\omega$. Thus $L_S\{f,g\} - \{L_S f, g\} - \{f, L_S g\}$ is a Hamilton function for $[S,[X_f,X_g]] - [[S,X_f],X_g] - [X_f,[S,X_g]] = 0$, and therefore constant on every connected component. Because L_S preserves $C^\infty_c(X)$ and $N(X)$, and because $[C^\infty_c(X), C^\infty_c(X)] = N(X)$, we see that $L_S\{f,g\} - \{L_S f, g\} - \{f, L_S g\}$ is an element of $N(X)$, and therefore zero.

The above reasoning shows that ψ_S is a 2-cocycle. If $S = X_h$ is hamiltonian (but not necessarily compactly supported), then $\psi_S = \delta\chi_h$, with $\chi_h(f) = \int_X hf\omega^{\wedge n}$. Indeed, $(L_{X_h} f)g\omega^{\wedge n} = -(L_{X_f} h)g\omega^{\wedge n} = -L_{X_f}(hg\omega^{\wedge n}) + h\{f,g\}\omega^{\wedge n}$, and the first term yields zero when integrated. Since $\mathrm{Sp}(X)/C^\infty(X) \simeq H^1(X,\mathbb{R})$ (de Rham cohomology where the cycles are *not* necessarily compactly supported), we obtain a map of presheaves $H^1(X,\mathbb{R}) \to H^2(N(X),\mathbb{R})$ that is explicitly given by

$$[\alpha] \mapsto \psi_{[\alpha]}(f,g) = \int_X \alpha(X_f) g\omega^{\wedge n}\,.$$

(We set $\alpha = i_S\omega$ and use $L_S f = i_S df = -i_S i_{X_f}\omega = (i_S\omega)(X_f)$.)

For $C^\infty_c(X) = N(X) \oplus \mathfrak{z}$, there are, in addition to the symplectic vector fields, the derivations which are zero on $N(X)$ and antisymmetric linear maps $d : \mathfrak{z} \to \mathfrak{z}$ on the centre. This yields a map of presheaves

$$\mu : H^1(X,\mathbb{R}) \oplus \wedge^2 H^{2n}_c(X,\mathbb{R})^* \to H^2_{LA}(C^\infty_c(X),\mathbb{R})\,.$$

3.1.2 Reduction to the local case for $C^\infty_c(X,\mathbb{R})$

We assume X to be connected, and choose a cover $\{U_i\}$ by open sets with star-shaped intersections. We forget the $\wedge^2 H^0_c$-part (which will not appear for connected sets anyway), and consider $\mu : \mathcal{F} \to \mathcal{S}$ with $\mathcal{F}(U) = H^1(U,\mathbb{R})$ and $\mathcal{S}(U) = H^2_{LA}(C^\infty_c(U),\mathbb{R})$.

The presheaf $\mathcal{F}$ is extremely simple: $\mathcal{F}(X) = H^1(X,\mathbb{R})$, and $\mathcal{F}(U_{i_1\ldots i_n}) = \{0\}$. Thus $\check{H}^{-1}\mathcal{F} = H^1(X,\mathbb{R})$ and $\check{H}^i\mathcal{F} = \{0\}$ for $i \geq 0$ simply because all chains are trivial. In particular, $\check{H}^0\mu$ is injective. We will show that also $\check{H}^{-1}\mu$ is injective.

$\check{H}^{-1}\mu$ is injective iff μ_X is injective on $\mathcal{F}(X)$, i.e. if $\psi_\alpha = \delta\chi$ implies that α is exact. In order to prove this, we construct a surjective map $\mu_X(\mathcal{F}(X)) \to \check{H}^1(X,\underline{\mathbb{R}})$. Since $H^1(X,\mathbb{R})$ (de Rham cohomology) is isomorphic to $\check{H}^1(X,\underline{\mathbb{R}})$ by de Rham's theorem, the leftmost map in

$$H^1(X,\mathbb{R}) \to \mu_X(\mathcal{F}(X)) \to \check{H}^1(X,\underline{\mathbb{R}}) \to 0 \tag{12}$$

cannot have a kernel, proving injectivity of μ_X.

The construction goes as follows. Every cocycle ψ_S can locally (on U_i) be written as $\delta\chi_{h_i}$, with $h_i \in C^\infty(U_i,\mathbb{R})$ such that $-i_S\omega = dh_i$. Clearly $d(h_i - h_j) = 0$ on U_{ij}, so ψ_S gives rise to a 1-cochain $c^1_{ij} = h_i - h_j$ with values in $\underline{\mathbb{R}}$, and one checks that $\delta c^1 = 0$. If $\psi_S = \delta\chi$, then certainly $\psi_S|_{U_i} = \delta\chi_{h_i} = \delta\chi|_{U_i}$ and $\delta(\chi_{h_i} - \chi_{U_i}) = 0$. Since $H^1(C^\infty_c(U_i),\mathbb{R})$ is one dimensional with generator $f \mapsto \int_{U_i} f$, we must have $(\chi_{h_i} - \chi_{U_i})(f) = \int c^0_i f\omega^{\wedge n}$ for some $c^0_i \in \mathbb{R}$. We thus have $\chi|_{U_i} = \chi_{h_i+c_i^0}$, and restricting to U_{ij}, we have $h_i + c_i^0 = h_j + c_j^0$, and thus $c_{ij}^1 = h_i - h_j = -(\delta c^0)_{ij}$. This shows that the class $[c^1] \in \check{H}^1(X, \underline{\mathbb{R}})$ depends only on the class of $[\psi_S]$. Since $[c^1]$ is precisely the image of $[i_S\omega] \in H^1(X, \mathbb{R})$ in $\check{H}^1(X, \underline{\mathbb{R}})$ under the isomorphism that comes from de Rham's theorem, the right hand map in (12) is surjective, so, as discussed, the left hand map must be injective.

Remark 3.4 *It would be surprising if the map described above would not just be the second page differential* $d_2^{-1,2}$.

Because we already know that $\check{H}^{-1}\mathcal{S} \simeq \check{H}^1(X, \underline{\mathbb{R}})$, this implies that $\check{H}^{-1}\mu$ is an isomorphism. Since $\check{H}^0\mu$ is injective, proposition 2.16 tells us that μ is an isomorphism of presheaves if and only if it is an isomorphism locally. We have proven

Lemma 3.5 *The map* μ *is an isomorphism of presheaves if and only if*

$$H^2_{LA}(C_c^\infty(U), \mathbb{R}) = \{0\}$$

for every open, star shaped neighbourhood U *in* $\mathbb{R}^{2n}$.

3.1.3 Conformal symplectic vector fields

A conformal symplectic vector field S is one that satisfies $L_S\omega = \lambda\omega$ with $\lambda \in C^\infty(X, \mathbb{R})$. Then $L_S f$ is a hamilton function for $[S, X_f] + \lambda X_f$, because $dL_S f = di_S df = -di_S i_{X_f}\omega = di_{X_f} i_S\omega = L_{X_f} i_S\omega - i_{X_f} di_S\omega = i_S L_{X_f}\omega + i_{[X_f,S]}\omega - i_{X_f}\lambda\omega = -i_{[S,X_f]+\lambda X_f}\omega$. Thus $L_S\{f,g\} - \{L_S f, g\} - \{f, L_S g\}$ is a Hamilton function for $X_g(\lambda)X_f - X_f(\lambda)X_g - \lambda[X_f, X_g]$.

Assume that λ is constant, $\lambda = c$. (E.g. $S = \sum_{i=1}^n x_i \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial y_i}$ with $c = 2$.) The operator $H := L_S + 1/2nc$ is skew symmetric w.r.t. the invariant bilinear form, and $H\{f,g\} - \{Hf, g\} - \{f, Hg\}$ is a hamilton function for $-3/2\lambda[X_f, X_g]$, and thus equal to $-3/2\{f,g\}$ up to a constant. We set $\psi_H(f,g) = \kappa(Hf, g)$. Then $\delta\psi_H(f,g,h) = \frac{n+2}{2}c\kappa(f, \{g,h\})$, so that the canonical third cohomology class is trivial.

Also, $[S,T]$ is symplectic if T is, because $L_{[S,T]}\omega = [L_S, L_T]\omega = -L_T c\omega = 0$. If $T = X_f$, then $[S,T]$ is even hamiltonian: $-i_{[S,X_f]}\omega = -L_S i_{X_f}\omega + i_{X_f} L_S\omega = L_S df - cdf = d(L_S f - cf)$. Thus $L_S - c$ is a derivation on $C_c^\infty(X)$.

3.2 Continuity of cocycles

Let $\psi : C^\infty(M) \times C^\infty(M) \to \mathbb{R}$ be a *continuous* (w.r.t. the topology induced by the seminorms $\|f\|^{\vec{\alpha},K} = \sup_K \|\partial_{\vec{\alpha}} f\|$ where K runs through the compact subsets of coordinate patches) cocycle. Since it is automatically local, the restriction to $C_c^\infty(M)$ of $\psi(f, \bullet) : C^\infty(M) \to \mathbb{R}$ is a distribution with compact support [Hel84, p. 240], denoted ψ_f. Because ψ is continuous, so is the map $f \mapsto \psi_f : C_c^\infty(M) \to C_c^\infty(M)'$. Clearly the restriction to $C_c^\infty(U)$ is continuous for every neighbourhood U of $x \in M$, so there are no points of discontinuity. According to Peetre's theorem [Pee60], there exists on every compact subset K of a coordinate patch a finite number of distributions $\phi^{\vec{\alpha}\vec{\beta}}$ such that $\psi(f,g) = \sum_{\vec{\alpha},\vec{\beta}} \phi^{\vec{\alpha},\vec{\beta}}(\partial_{\vec{\alpha}} f \partial_{\vec{\beta}} g)$ for all f, g with support in K.

Any distribution ϕ on K takes the shape $\phi(f) = (-1)^{|\vec{\alpha}|} \int_K F(x)\partial_{\vec{\alpha}} f(x)dx$ with F continuous. We can perform integration by parts, to make sure F is C^1 (or in fact C^n) raising the degree of $\vec{\alpha}$. All in all, we may write

$$\psi_K(f, g) = \sum_{\vec{\alpha},\vec{\beta}} \int_K F^{\vec{\alpha},\vec{\beta}} \partial_{\vec{\alpha}} f \partial_{\vec{\beta}} g dx$$

for all f, g with support in K, where the $F^{\vec{\alpha},\vec{\beta}}$ can be chosen C^k.

Since ψ is antisymmetric, we have $\psi(f, g)+\psi(g, f) = 0$, i.e. (with $f_{\vec{\alpha}} := \partial_{\vec{\alpha}} f$)

$$\int_K (F^{\vec{\alpha},\vec{\beta}} + F^{\vec{\beta},\vec{\alpha}}) f_{\vec{\alpha}} g_{\vec{\beta}} = 0\,.$$

We can therefore replace $F^{\vec{\alpha},\vec{\beta}}$ by $\frac{1}{2}(F^{\vec{\alpha},\vec{\beta}} - F^{\vec{\beta},\vec{\alpha}})$ without changing ψ, and we assume without loss of generality that $F^{\vec{\alpha},\vec{\beta}}$ is antisymmetric.

Suppose that K is equipped with Darboux coordinates and consider the equation $\delta\psi = 0$, written as

$$\int \left(F^{\vec{\alpha},\vec{\beta}} \Omega^{\sigma\tau}\right) \left(\partial_{\vec{\alpha}} f \partial_{\vec{\beta}}(g_\sigma h_\tau) + \partial_{\vec{\alpha}} g \partial_{\vec{\beta}}(h_\sigma f_\tau) + \partial_{\vec{\alpha}} h \partial_{\vec{\beta}}(f_\sigma g_\tau)\right) = 0$$

Let $\Omega \subset \mathbb{R}^{2n}$ be an open subset, and let Let $\psi : C_c^\infty(\Omega) \times C_c^\infty(\Omega) \to \mathbb{R}$ be continuous and local: $\psi(f, g) = 0$ if f and g have disjoint support.

Consider the map $C_c^\infty(\Omega) \to D(\Omega) : f \mapsto \psi(f, \bullet)$, and denote the distribution $\psi(f, \cdot\,)$ by ψ_f. Since $\mathrm{Supp}(\psi_f) \subseteq$

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