Notes on Defects & String Groups

Version 2.0

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Abstract

We show that the 2-group model for the string group in the paper [DH12] of Christopher Douglas and André Henriques is isomorphic to the Stolz-Teichner model described in [ST04].

Beware! This is a draft version which has not been debugged, and one would do well not to take anything here on faith. I've tried to be explicit rather than elegant at every turn, which results in a level of detail that is probably excruciating for grown-up mathematicians.

In section 1, we spend a few words on free fermionic field on the circle. In section 2, we loosely follow [Was98] (which we cannot apply directly because our free fermions are Majorana and not Dirac fermions), and calculate the modular operators. This is rather subtle, but probably well known. The gauge group for these Majorana fermions is $SO(n)$ rather than $U(n)$. In section 3, we look at the projective representations of the corresponding loop groups. It contains proposition 12 (cf. [PS86]), which says that the strong topology on the von Neumann algebra of field operators is the thing that detects the holonomy of a loop. Without this, these notes would collapse as a plumb pudding. In section 4, we define defects and sectors for the free fermionic field, and calculate fusion products of these sectors. The trick is to use the identification of the (abstract) bimodule $L^2(\mathcal{A}(I))$ with the (concrete) Fock space $\mathcal{F}$, in which the Tomita-Takesaki involution J is simply a reflection. In this way, a bit of toil provides a very explicit description of the sectors that live over the defects in which we're interested. In section 5, we reshuffle this information to yield the statement that the weak 2-group $G(V)$ described by Douglas and Henriques in [DH12] is isomorphic (as a weak 2-group) to the (strict) model described by Stephan Stolz and Peter Teichner in [ST04]. The point is not so much that these models are isomorphic; as the name suggests, weak isomorphism is not a very strong notion. The point is that the isomorphism is nice. In section 8, I take the liberty of presenting you, dear reader, with some questions rather than answers.

1 Free Majorana Fermions

Following [DH12], we briefly describe free fermions, restricting ourselves to subintervals of the circle $S^1 \subset \mathbb{R}^2$. Classically, a fermion on S^1 is an element of $L^2(\mathrm{Moe})$, where $\mathrm{Moe} \to S^1$ is the (real!) Möbius band. If we have d fermionic fields, the classical fields are $L^2(\mathrm{Moe} \otimes_{\mathbb{R}} V)$ with $V \simeq \mathbb{R}^d$ some (real) vector space with inner product. We denote $\mathrm{Moe} \otimes_{\mathbb{R}} V$ by Moe^V. The embedding $S^1 \to \mathbb{R}^2$ induces the metric $d\phi$ and SO(2)-action on S^1, and also (by the trivial spin structure on $\mathbb{R}^2$) a Spin(2)-action on Moe^V that covers the SO(2)-action on S^1.

This Spin(2)-action determines a polarisation, i.e. a complex structure on the (real!) Hilbert space $L^2(\mathrm{Moe}^V)$, on which the inner product is given by $\langle f, g\rangle := \frac{1}{2\pi}\int_0^{2\pi}(f(\phi), g(\phi))d\phi$. One identifies $L^2(\mathrm{Moe}^V \otimes_{\mathbb{R}} \mathbb{C})$ with $l^2(\mathbb{Z}+\frac{1}{2})\otimes V$ by Fourier transform $z^{n+\frac{1}{2}} \otimes v \mapsto \delta_{-n} \otimes v$, so that $L^2(\mathrm{Moe}^V)$ corresponds with

$$\{f \in l^2_{\mathbb{C}}(\mathbb{Z} + \tfrac{1}{2}) \otimes_{\mathbb{R}} V \,;\, f(-n) = \overline{f(n)}\}.$$

(This 'diagonalises' the Spin(2)-action in the sense that $[[\delta_n + \delta_{-n}, i\delta_n - i\delta_{-n}]] \otimes V$ is the real 2d-dimensional isotypical component with weight $2n + 1$.) If $D = -2i\frac{d}{d\phi}$ is the (selfadjoint) generator of the Spin(2)-action, diagonalised on $l^2(\mathbb{Z}+\frac{1}{2})$ as $n+\frac{1}{2} \mapsto 2n+1$, then we define the complex structure by $J := i\,\mathrm{sg}(D)$, which is well defined because $0 \notin \mathrm{Spec}(D)$. Explicitly, $J(f)(n) = i\,\mathrm{sg}(n)f(n)$ for all $n \in \mathbb{Z}+\frac{1}{2}$, and we see that J, unlike D, restricts to the real Hilbert space.

Given the real Hilbert space $L^2(\mathrm{Moe}^V)$ with complex structure J, there are two ways to define the Fermionic Fock space $\mathcal{F}$: with or without antiparticles. Following [DH12] (and and in contrast to [Was98]) we take version without antiparticles (corresponding to majorana fermions), because this leads to a SO(n) gauge group (in contrast to the SU(n) gauge group in [Was98].) Thus

$$\mathcal{F} := \bigwedge_J L^2(\mathrm{Moe}^V) \simeq \bigwedge P_+ L^2(\mathrm{Moe}^V \otimes_{\mathbb{R}} \mathbb{C})\,,$$

with $P_+ := \chi_{\mathbb{R}^+}(D)$ the projection onto the positive energy part. (The $\mathbb{C}$-linear isomorphism $P_+L^2(\mathrm{Moe}^V \otimes_{\mathbb{R}} \mathbb{C}) \to L^2(\mathrm{Moe}^V)$ is given by $e^{in\phi} \otimes v \mapsto \cos(n\phi) \otimes v$ for $n > 0$ in $\mathbb{Z} + \frac{1}{2}$, $v \in V$). The 'with antiparticles' (Dirac fermion) version would have been

$$\mathcal{F} := \bigwedge_J L^2(\mathrm{Moe}^V \otimes_{\mathbb{R}} \mathbb{C}) \simeq \bigwedge P_+ L^2(\mathrm{Moe}^V \otimes_{\mathbb{R}} \mathbb{C}) \hat{\otimes} \bigwedge P_- L^2(\mathrm{Moe}^V \otimes_{\mathbb{R}} \mathbb{C})^*\,.$$

I'll write $\mathrm{Moe}^{V,\mathbb{C}}$ for $\mathrm{Moe} \otimes_{\mathbb{R}} V \otimes_{\mathbb{R}} \mathbb{C}$.

If $I \subseteq J \subseteq S^1$, then the injective isometry $L^2(\mathrm{Moe}^V|_I) \hookrightarrow L^2(\mathrm{Moe}^V|_J)$ yields a natural inclusion $\mathrm{Cl}(L^2(\mathrm{Moe}^{V,\mathbb{C}}|_I)) \hookrightarrow \mathrm{Cl}(L^2(\mathrm{Moe}^{V,\mathbb{C}}|_J))$ of the respective Clifford algebras. The norm closure $\mathrm{CAR}(I)$ of $\mathrm{Cl}(L^2(\mathrm{Moe}^{V,\mathbb{C}}|_I))$ depends only on the Hilbert space structure, so we also get an inclusion $\mathrm{CAR}(I) \hookrightarrow \mathrm{CAR}(J)$.

We define $\mathcal{A}(I) := \mathrm{CAR}(I)''$, the closure of the appropriate Clifford algebra w.r.t. the weak topology induced by the vacuum state in $\mathcal{F}$. $I \mapsto \mathcal{A}(I)$ is a precosheaf of vN-algebras. The weak closure of $\mathrm{CAR}(S^1)$ is simply $B(\mathcal{F})$, but the situation for $\mathrm{CAR}(I)$ with $I^c := S^1 - \overline{I}$ open and nonempty is a bit more subtle, and involves modular operators.

2 Modular Operators

If $\mathcal{A}$ is a v.N. algebra acting on a Hilbert space $\mathcal{F}$ with separating cyclic vector Ω, then the operator $S : \mathcal{A}\Omega \to \mathcal{F}$ defined by $a\Omega \mapsto a^*\Omega$ extends to a closed, antilinear, unbounded operator, which has polar decomposition $S = J\Delta^{\frac{1}{2}} = \Delta^{-\frac{1}{2}}J$. (With $\Delta \geq 0$ positive linear and J anti-unitary, $\langle J\xi, J\eta\rangle = \overline{\langle \xi, \eta\rangle}$.)

2.1 Example 0: matrices

Let $\mathcal{A} = B(\mathbb{C}^n)$ be the v.N. algebra of $n \times n$-matrices. A normal state is faithful if and only if it is given by $\rho(a) = \mathrm{tr}(Ra)$ for some invertible density matrix $R > 0$ in $\mathcal{A}$. We consider the GNS-representation $L^2_\rho(\mathcal{A})$, which is the closure of $\mathcal{A}$, equipped with the inner product $\langle a, b\rangle = \rho(a^*b)$. This has a natural left *-action of $\mathcal{A}$, given by $a[\xi] = [a\xi]$, and a natural right *-action given by $[\xi]a := [\xi R^{\frac{1}{2}} a R^{-\frac{1}{2}}]$.

Proposition 1. *This is a right *-action:* $\langle [\xi], [\eta]a\rangle = \langle [\xi]a^*, [\eta]\rangle$.

Proof. It is clearly a right action, the only thing to check is the involution.

$$\begin{aligned}\langle [\xi], [\eta]a\rangle &= \mathrm{tr}(R\xi^*\eta R^{\frac{1}{2}} a R^{-\frac{1}{2}}) = \mathrm{tr}(\eta R^{\frac{1}{2}} a R^{\frac{1}{2}} \xi^*) = \mathrm{tr}(\eta R(R^{-\frac{1}{2}} a R^{\frac{1}{2}} \xi^*)) \\ &= \mathrm{tr}(R(\xi R^{\frac{1}{2}} a^* R^{-\frac{1}{2}})^*\eta) = \langle [\xi]a^*, [\eta]\rangle\,.\end{aligned}$$

□Note that $[\xi]a = [\xi a]$ would define a right action of algebras, but not of *-algebras! We calculate the modular operators for $(L^2_\rho(\mathcal{A}), \Omega)$.

Proposition 2. *We have* $S[\xi] = [\xi^*]$, $\Delta([\xi]) = [R\xi R^{-1}]$, $J([\xi]) = [R^{\frac{1}{2}}\xi^* R^{-\frac{1}{2}}]$. *In particular, the right action is given by* $[\xi]a = Ja^*J[\xi]$.

Proof. Check the formula for Δ, sandwiched between arbitrary vectors:

$$\begin{aligned}\langle \Delta[\xi], [\eta]\rangle &= \langle S^\dagger S[\xi], [\eta]\rangle = \overline{\langle S[\xi], S[\eta]\rangle} = \overline{\mathrm{tr}(R\xi\eta^*)} \\ &= \mathrm{tr}(\eta\xi^* R) = \mathrm{tr}(R(R\xi R^{-1})^*\eta) = \langle [R\xi R^{-1}], [\eta]\rangle\,.\end{aligned}$$

Thus $\Delta^{\frac{1}{2}}([\xi]) = [R^{\frac{1}{2}}\xi R^{-\frac{1}{2}}]$, and the formula $J[\xi] = [R^{\frac{1}{2}}\xi^* R^{-\frac{1}{2}}]$. for J follows from $J = \Delta^{\frac{1}{2}} S$. One easily checks the formula for the right action:

$$Ja^*J[\xi] = J([a^* R^{\frac{1}{2}}\xi^* R^{-\frac{1}{2}}]) = [R^{\frac{1}{2}}(R^{\frac{1}{2}}\xi^* R^{-\frac{1}{2}})^* a R^{-\frac{1}{2}}] = [\xi R^{\frac{1}{2}} a R^{-\frac{1}{2}}]\,.$$

□Since $0 < R \leq 1$, we can always write $R = e^{-H}$ with $H \geq 0$. The modular flow $\sigma^R_t(x) = \Delta^{it} x \Delta^{-it}$ is then given by $e^{itH} x e^{-itH}$, the time evolution for the Hamiltonian w.r.t. which R is the thermal equilibrium state.

Remark If $\mathcal{A} \subset B(\mathcal{H})$ is a v.N. algebra with faithful normal state given by $\rho(a) = \mathrm{tr}(Ra)$ with $R \in \mathcal{A}$ a positive trace class operator, then essentially nothing changes.

2.2 Example 1: Clifford algebras

Let $\mathcal{H}_\mathbb{R}$ be a real Hilbert space. Its complexification $\mathcal{H}_\mathbb{C}$ then comes with an antilinear involution $j : v \mapsto \overline{v}$, a $\mathbb{C}$-bilinear form $B(v, w)$ extending the real inner product, and the positive definite Hermitean form $\langle v, w\rangle = B(\overline{v}, w)$. For now, we assume that $\mathcal{H}_\mathbb{R}$ is of finite, even dimension $2n$. This makes things easier, but – contrary to popular belief – not trivial.

Let $\mathcal{H}_\mathbb{C} = V_+ \oplus V_-$ be a polarisation, i.e. an orthogonal direct sum with $\overline{V}_+ = V_-$. We denote by $P_\pm$ the orthogonal projection onto $V_\pm$. We can then form the spinor representation $\mathcal{F} = \mathrm{Cl}(\mathcal{H}_\mathbb{C})/\mathrm{Cl}(\mathcal{H}_\mathbb{C})V_-$, i.e. the $\mathrm{Cl}(\mathcal{H}_\mathbb{C})$-representation defined by left multiplication, quotiented by the smallest subrepresentation containing $V_- \subset \mathrm{Cl}(\mathcal{H}_\mathbb{C})$. We denote $\Omega := [\mathbf{1}]$. The Hilbert space $\mathrm{Cl}(V_+) = \bigwedge V_+$ naturally includes into $\mathcal{F}$, and one shows that this map is surjective, so that $\mathcal{F}$ receives a Hilbert space structure with respect to which the left $\mathrm{Cl}(\mathcal{H}_\mathbb{C})$-action is unitary.

Let $\mathcal{H}_\mathbb{C} = \mathcal{H}_+ \oplus \mathcal{H}_-$ be a second orthogonal decomposition into subspaces, this time closed under conjugation: $\overline{\mathcal{H}}_+ = \mathcal{H}_+$ and $\overline{\mathcal{H}}_- = \mathcal{H}_-$. Denote by $Q_\pm$ the orthogonal projection onto $\mathcal{H}_\pm$. We show that $\mathcal{F}$ is the GNS representation for $\mathrm{Cl}(\mathcal{H}_+)$ w.r.t. the vacuum state for $P_\pm$.

Proposition 3. *If the projections $P_\pm$ and $Q_\pm$ are in general position, i.e. $V_\pm \cap \mathcal{H}_\pm = \{0\}$ for all 4 choices of sign, then Ω is a cyclic and separating vector for the $\mathrm{Cl}(\mathcal{H}_+)$ action on $\mathcal{F}$.*

Proof. The projections are in general position if and only if the operators $T_+ := (P_+ - Q_+)^2$ and $T_- := (P_+ - Q_-)^2$ are invertible. One checks that $T_+ + T_- = \mathbf{1}$ holds, and moreover the reflection formulæ

$$\begin{aligned}
P_+T_+ &= P_+Q_-P_+ = T_+P_+ \quad , \quad Q_+T_+ = Q_+P_-Q_+ = T_+Q_+\\
P_-T_+ &= P_-Q_+P_- = T_+P_- \quad , \quad Q_-T_+ = Q_-P_+Q_- = T_+Q_-\\
P_+T_- &= P_+Q_+P_+ = T_-P_+ \quad , \quad Q_+T_- = Q_+P_+Q_+ = T_-Q_+\\
P_-T_- &= P_-Q_-P_- = T_-P_- \quad , \quad Q_-T_- = Q_-P_-Q_- = T_-Q_- \,.
\end{aligned}$$

In particular, $[T_\pm, P_\pm] = [T_\pm, Q_\pm] = 0$, so $T_\pm$ respect $V_\pm$ and $\mathcal{H}_\pm$.

We show that for every $v \in V_+$, there exist unique $w \in V_-$ and $f \in \mathcal{H}_+$ so that $f = v + w$. Indeed, $f = Q_+T_-^{-1}P_+v$ and $w = P_-f$ do the job:

$$P_+f = P_+Q_+T_-^{-1}P_+v = T_-^{-1}P_+Q_+P_+v = T_-^{-1}T_-P_+v = v\,.$$

Uniqueness is clear: $P_+ : \mathcal{H}_+ \to V_+$ is injective because $V_+ \cap \mathcal{H}_+ = \{0\}$. In the same vein, for any $w \in V_-$, the unique $f \in \mathcal{H}_+$ and $v \in V_+$ such that $v + w = f$ are given by $f = Q_+T_+^{-1}P_-w$ and $v = P_+f$.

This ensures that Ω is cyclic for $\mathrm{Cl}(\mathcal{H}_+)$. Indeed, assume with induction that all the states in $\wedge^n V_+$ are in $\mathrm{Cl}(\mathcal{H}_+)\Omega$. Let $v \in V_+$ and $\xi = \psi(v)\xi_0 \in \wedge^{n+1}V_+$. Write $f = v+w$ with $f \in \mathcal{H}_+$ and $w \in V_-$. Then $\xi = \psi(v)\xi_0 = \psi(f)\xi_0 - \psi(w)\xi_0$, and since $\psi(w)\xi_0$ is in $\wedge^{n-1}V_+$ (push $\psi(w)$ to the right untill it hits Ω while keeping track of the commutators), we have $\xi \in \mathrm{Cl}(\mathcal{H}_+)\Omega$. Similarly, Ω is cyclic for $\mathrm{Cl}(\mathcal{H}_-)$, so because $\mathrm{Cl}(\mathcal{H}_+)$ and $\mathrm{Cl}(\mathcal{H}_-)$ are each other's graded commutant, Ω is also separating for $\mathrm{Cl}(\mathcal{H}_+)$. $\square$

2.2.1 Modular operators for Clifford algebras

An orthogonal transformation $u \in \mathcal{O}(\mathcal{H}_{\mathbb{R}})$ can be extended to a unitary u or antiunitary $\overline{u}$ operator on $\mathcal{H}_{\mathbb{C}}$ that commutes with complex conjugation j, and all unitary or antiunitary operators commuting with j are of this form. The unitary operator induces a linear *-automorphism of $\mathrm{Cl}(\mathcal{H}_{\mathbb{C}})$ because it preserves the bilinear form, $B(uf, ug) = B(f, g)$, whereas the antiunitary operator induces an antilinear *-automorphism of $\mathrm{Cl}(\mathcal{H}_{\mathbb{C}})$ because $B(uf, ug) = \overline{B(f, g)}$.

If, moreover, u maps V_+ to V_+ and V_- to V_-, then α_u induces a linear (or antilinear) map on $\mathcal{F} = \mathrm{Cl}(\mathcal{H}_{\mathbb{C}})/\mathrm{Cl}(\mathcal{H}_{\mathbb{C}}) \cdot V_-$. It is given by $\Lambda(u) : \psi(f_1) \ldots \psi(f_n)\Omega \mapsto \psi(uf_1) \ldots \psi(uf_n)\Omega$, regardless whether u is unitary or antiunitary. Unitary (antiunitary) maps that respect both $P_\pm$ and j correspond to unitary (antiunitary) operators on V_+, and the 'canonical second quantisation' $U_\pm(V_+) \to U_\pm(\mathcal{F})$ is a group homomorphism. The Clifford transposition $\tau : \psi(f_1) \ldots \psi(f_n) \mapsto \psi(f_1) \ldots \psi(f_n)$ obviously doesn't preserve $\mathrm{Cl}(\mathcal{H}_{\mathbb{C}})V_-$, but through the identification $\mathcal{F} \simeq \bigwedge V_+$, it still defines a unitary operator on $\mathcal{F}$. One checks that it is given by $\tau = \kappa^{-1}\Lambda(i\mathbf{1})$, with κ the Klein transformation. The Klein transformation κ multiplies by 1 on the even and by i on the odd part. It has the property that a and b supercommute, $a \cdot b = (-1)^{|a||b|} b \cdot a$, if and only if a and $\kappa b \kappa^{-1}$ commute. For antiunitary operators, we define $\tilde{\Lambda}(u) := \tau \circ \Lambda(u) = \kappa^{-1}\Lambda(iu)$.

We extend this procedure to arbitrary linear (or antilinear) operators σ on $\mathcal{H}_{\mathbb{C}}$ that respect j and $P_\pm$. Write $\sigma = u\delta^{1/2}$ as a product of an (anti)-unitary and a positive operator, both respecting j and $P_\pm$ (this can essentially be done on V_+). Then $t \mapsto \delta^{it}$ is a 1-parameter group of unitaries, where $\delta^{it} = e^{-itH}$ for $H = -\log(\delta)$. Consequently, $\dot{\Lambda}(H) := i\frac{d}{dt}|_0 \Lambda(e^{-itH})$ is selfadjoint, and we define $\Lambda(\delta^{1/2}) := \exp(-\frac{1}{2}H)$, and $\Lambda(\sigma) := \Lambda(u)\Lambda(\delta^{1/2})$. Since our Hilbert spaces are finite dimensional, the map $z \mapsto \psi(\delta^{iz} f_1) \ldots \psi(\delta^{iz} f_n)\Omega$ is well defined and holomorphic on $\mathbb{C}$, so that $\Lambda(\delta^{1/2})$ is simply given by its induced action on $\mathcal{F} = \bigwedge V_+$, $\Lambda(\delta^{1/2})\psi(f_1) \ldots \psi(f_n)\Omega = \psi(\delta^{1/2} f_1) \ldots \psi(\delta^{1/2} f_n)$. In particular, Λ extends to a homomorphism on $\mathrm{Gl}_\pm(V_+)$.

Proposition 4. *The modular operators for the representation of* $\mathrm{Cl}(\mathcal{H}_+)$ *on* $\mathcal{F}$ *defined by* $P_\pm$ *are given by* $S = \kappa^{-1}\Lambda(i\sigma)$, $\Delta^{1/2} = \Lambda(\delta^{1/2})$ *and* $J = \kappa^{-1}\Lambda(iu)$, *where* $\sigma = u\delta^{1/2} = \delta^{-1/2}u$ *is the polar decomposition of the antilinear operator*

$$\sigma := j\left(\frac{P_+Q_+P_-}{T_+} + \frac{P_-Q_+P_+}{T_-}\right),$$

given by

$$\delta^{1/2} = \sqrt{\frac{T_+}{T_-}}P_+ + \sqrt{\frac{T_-}{T_+}}P_-$$

and

$$u = j\left(\frac{P_-Q_+P_+}{\sqrt{T_+T_-}} + \frac{P_+Q_+P_-}{\sqrt{T_+T_-}}\right),$$

where $T_+ := (P_+ - Q_+)^2$ *and* $T_- := (P_+ - Q_-)^2$ *commute with* $P_\pm$*,* $Q_\pm$ *and satisfy* $T_+ + T_- = \mathbf{1}$*. The operator* $\delta^{1/2}$ *posesses a basis of eigenvectors* $v_\phi^\pm$ *such that* $v_\phi^+ \in V_+$*,* $v_\phi^- \in V_-$*, the eigenvalues for* $v_\phi^\pm$ *under* T_+ *and* T_- *are* $\sin^2(\phi)$ *and* $\cos^2(\phi)$ *respectively,* $\delta^{1/2}v_\phi^+ = |\tan(\phi)|v_+$*, and* $\delta^{1/2}v_\phi^- = |\tan(\phi)|^{-1}v_-$*, where* ϕ *is the angle between* V_+ *and* $\mathcal{H}_+$ *(or* V_- *and* $\mathcal{H}_-$*) along* v_+^ϕ *(or* v_-^ϕ*). For one angle* ϕ*, the 2-dimensional vector space* $[[v_\phi^+, v_\phi^-]]$ *is therefore closed under* $P_\pm$ *and* $Q_\pm$*. The operators* $\delta^{1/2}$ *and* u *commute with* j*,* $\delta^{1/2} > 0$ *and* ju *is the unitary involution exchanging* $v_\phi^+ \leftrightarrow v_\phi^-$*.*

Proof. Let σ be the antilinear operator $\sigma := j \circ (P_+Q_+T_+^{-1}P_- + P_-Q_+T_-^{-1}P_+)$. $j\sigma$ maps $v \in V_+$ to the unique $w \in V_-$ such that $v + w \in \mathcal{H}_+$, and $w \in V_-$ to the unique $v \in V_+$ with the same property. In particular, it is the identity on $\mathcal{H}_+$, and it squares to one, $\sigma^2 = 1$. Since $jP_\pm = P_\mp j$, $jQ_\pm = Q_\pm j$ and $jT_\pm = T_\mp j$, we have $[\sigma, j] = 0$, $[\sigma, P_\pm] = 0$ and $\sigma Q_+ = jQ_+$. Let $\sigma = u\delta^{1/2}$ be the polar decomposition of σ. Since $\sigma^2 = 1$, $\sigma^\dagger\sigma = \delta$ is the inverse of $\sigma\sigma^\dagger = u\delta u^\dagger$. This shows that $\sigma = \delta^{1/2}u = u\delta^{-1/2}$, and thus $u^2 = \sigma^2 = 1$.

Its second quantisation $\tilde\Lambda(\sigma) = \kappa^{-1}\Lambda(i\sigma)$ is antilinear on $\mathcal{F}$, and for $f_i \in \mathcal{H}_+$, it maps (recall that $\sigma Q_+ = jQ_+$) the vector $\psi(f_1)\ldots\psi(f_n)\Omega$ to $\psi(\overline{f}_n)\ldots\psi(\overline{f}_1)\Omega$. It therefore agrees with the operator $S : a\Omega \mapsto a^\dagger\Omega$ defined on the abstract GNS-representation of $\mathcal{H}_+$. Because $\tilde\Lambda(u\delta^{1/2}) = \tilde\Lambda(u)\Lambda(\delta^{1/2})$ is again a polar decomposition, we have $J = \tilde\Lambda(u) = \kappa^{-1}\Lambda(iu)$ and $\Delta = \Lambda(\delta)$.

We proceed to calculate the polar decomposition. Using the reflection formulæ in the previous proposition, one calculates $\delta = \sigma^\dagger\sigma = T_+T_-^{-1}P_+ + T_-T_+^{-1}P_-$, and notes that $[j, \delta] = 0$. One then calculates $u = \delta^{1/2}\sigma$, yielding the above result.

This is best understood when we diagonalise P_+, P_-, T_+, T_- and $\delta^{1/2}$. Let $v_+ \in V_+$ be a simultaneous eigenvector. Then $T_-v_+ = P_+Q_+v_+ = c^2v_+$, so that $c^2 = \langle v, Q_+v_+\rangle = \cos^2(\phi)$, with ϕ the angle between V_+ and $\mathcal{H}_+$ along the vector v. Similarly, the eigenvalue of T_+ for $v \in V_+$ is $\sin^2(\phi)$, so the eigenvalue for $\delta^{1/2}$ is $|\tan(\phi)|$. Because ju is invertible, exchanges V_+ and V_-, commutes with T_+ and T_- and satisfies $(ju)\delta^{1/2} = \delta^{-1/2}(ju)$, the vector $v_- := juv_+$ in V_- is again a simultaneous eigenvector, this time with eigenvalue $|\tan(\phi)|^{-1}$ for $\delta^{1/2}$. □

2.3 Example 2: Infinite dimensional Clifford Algebras

We now move to the realm of infinite dimensional Clifford algebras, i.e. we drop the assumption that $\mathcal{H}_\mathbb{R}$ is finite dimensional. In this case, proposition 4 remains valid (except for the 'basis of eigenvectors' part of course), but there are some details to look after.

We now require that the polarisations are by closed subspaces, and that the operators T_+ and T_- are injective, but do not necessarily have a bounded inverse. (0 is allowed to be a bounary point in the spectrum). We define $\mathcal{F}$ as the closure of $\mathrm{Cl}(\mathcal{H}_\mathbb{C})/\mathrm{Cl}(\mathcal{H}_\mathbb{C}) \cdot V_-$. The C*-algebra generated by $\mathrm{Cl}(\mathcal{H}_\mathbb{C})$ is the CAR-algebra. On unitary and antiunitary operators, canonical quantisation is well defined and a group homomorphism, and for unbounded positive operators we can still define $\Lambda(\delta^{1/2}) = e^{-\frac{1}{2}\dot{\Lambda}(H)}$, with $H = -\log(\delta)$. For $\sigma = u\delta^{1/2}$, the equation

$$\psi(\sigma\xi_n)\ldots\psi(\sigma\xi_1)\Omega = \tilde{\Lambda}(u)\Lambda(\delta^{1/2})\psi(\xi_1)\ldots\psi(\xi_n)\Omega$$

now holds (and makes sense!) for $\xi_1 \ldots \xi_n \in \mathcal{H}_\mathbb{C}^\infty$, i.e. vectors such that $t \mapsto \delta^{it}\xi$ is smooth. By a theorem of Stone and Gårding, $\mathcal{H}_\mathbb{C}^\infty$ is dense in $\mathcal{H}_\mathbb{C}$, so $\bigwedge \mathcal{H}_\mathbb{C}^\infty$ is dense in $\mathcal{F}$. The above equation then holds on the domain of $\Lambda(\delta^{1/2})$ by continuity. Rather than diagonalising T_+, one uses its spectral measure and the equation $T_+ + T_- = 1$ to define the operators σ, $\delta^{1/2}$ and u, and because $\sigma Q_+ = jQ_+$, the domain of δ (which is the domain of σ) containt $\mathcal{H}_+$. The equality $\psi(\overline{f}_n)\ldots\psi(\overline{f}_1)\Omega = \tilde{\Lambda}(u)\Lambda(\delta^{1/2})$ follows, showing that $S = J\Delta^{1/2}$ with $J = \tilde{\Lambda}(u)$ and $\Delta^{1/2} = \Lambda(\delta^{1/2})$ with u and δ as in proposition 4 are the modular operators.

2.3.1 Modular operators for Majorana Fermions

Roughly following the second proof of theorem 14 in [Was98], we implement the above in the situation $\mathcal{H}_\mathbb{R} = \Gamma_{L^2}(\mathrm{Moe} \otimes V)$, where the vacuum polarisation $\mathcal{H}_\mathbb{C} = V_+ \oplus V_-$ is given by the projections P_+ and P_- on the positive (negative) part of the spectrum of $-2i\frac{d}{d\phi}$. The Fock space $\mathcal{F}$ is thus the closure of $\bigwedge V_+$. The other polarisation $\mathcal{H}_\mathbb{C} = \mathcal{H}_+ \oplus \mathcal{H}_-$ is given by an interval $I \subset S^1$. We require that both I and its complement I^c have nonempty interior. Then $\mathcal{H}_+ = \{s \in \mathcal{H}_\mathbb{C}\,;\ s|_{I^c} = 0\}$ and $\mathcal{H}_- = \{s \in \mathcal{H}_\mathbb{C}\,;\ s|_I = 0\}$, and the corresponding projections are Q_+ and Q_-. We set $js(\phi) = \overline{s}(\phi)$, so that $jQ_+ = Q_+j$ and $jP_+ = P_-j$.

Remark 1. Since the vector space V does not play any role in the determination of the modular operators, we leave it aside until the end of this section, and just keep in mind that everything we do should be tensored with $V \simeq \mathbb{R}^n$.

If $\tilde{S}^1 \to S^1$ is the twofold cover of the circle (which is an $O(2)$-torsor and thus a Pin structure), then a section of Moe is a smooth function $s : \tilde{S}^1 \to \mathbb{R}$ such that $s(-e^{i\chi}) = -s(e^{i\chi})$. We identify $C^\infty(S^1)$ with $\Gamma(\mathrm{Moe})$ isometrically by associating to $g \in C^\infty(S^1)$ the section $s(e^{i\chi}) = e^{i\chi}g(e^{2i\chi})$. On $L^2(S^1)$, we have $D = 1 - 2i\frac{d}{d\phi}$, so that $V_+ = \mathrm{Hol}(\Delta_+)$ is the space of holomorphic functions on the complex disc Δ_+ that extend to L^2-functions on S^1. Its orthogonal complement $V_- = \mathrm{Hol}_0(\Delta_-)$ is the space of holomorphic functions on $\Delta_- = \mathbb{C}P^1 - \overline{\Delta}_+$ which extend to L^2-functions on S^1, and that furthermore satisfy $g(\infty) = 0$. (So $1 \in V_+$, $1 \notin V_-$.)

The spaces $\mathcal{H}_+$ and $\mathcal{H}_-$ are the L^2-functions on S^1 that disappear in I^c and I respectively. In the following, we will take I to be the right semicircle $I = \{e^{i\phi}\,;\, \phi \in [-\pi/2, \pi/2]\}$.

Definition 1. *The antilinear involution j takes the shape $jg(e^{i\phi}) = e^{-i\phi}\overline{g}(e^{i\phi})$. It corresponds to the map $\mathrm{Hol}(\Delta_+) \oplus \mathrm{Hol}_0(\Delta_-)$ that maps $g(z)$ to $jg(z) = z^{-1}\overline{g}(\overline{z}^{-1})$. There are two possible ways to lift the left-right reflection $e^{i\phi} \mapsto -e^{-i\phi}$ from S^1 to $\tilde{S}^1$, namely $s_\pm(e^{i\chi}) = \pm ie^{-i\chi}$. They yield the involutions $s_\pm g(z) = \pm iz^{-1}g(-z^{-1})$. Similarly, the up-down flips are given by $F_\pm g(e^{i\phi}) = \pm e^{-i\phi}g(e^{-i\phi})$.*

2.3.2 Cayley Transform

In order to diagonalise $T_+ = (P_+ - Q_+)^2$, we return to the origin of S^1 as the conformal compactification of the real line. By means of (i times) the Cayley transform Γ, i.e. the Moebius transformation

$$\Gamma(z) = i\frac{z-i}{z+i}, \qquad \Gamma^{-1}(w) = -i\frac{w+i}{w-i}\,,$$

we map the unit disc $\Delta_+ = \{z \in \mathbb{C}\,;\, |z| < 1\}$ to the lower half plane $\mathbb{H}_- = \{z \in \mathbb{C}\,;\, \mathrm{Im}(z) < 0\}$ and vice versa. The Cayley transform induces unitary tranformations on the boundaries of these domains, $U_\Gamma : L^2(S^1, d\phi) \to L^2(\mathbb{R}, dx)$ and $U_\Gamma^{-1} : L^2(\mathbb{R}, dx) \to L^2(S^1, d\phi)$, by

$$(U_\Gamma g)(x) = \frac{\sqrt{\pi}}{x-i}g\left(-i\frac{x+i}{x-i}\right) \quad (U_\Gamma^{-1}f)(e^{i\phi}) = \frac{2\sqrt{\pi}}{e^{i\phi}+i}f\left(i\frac{e^{i\phi}-i}{e^{i\phi}+i}\right)\,.$$

(We used $e^{i\phi} = -i\frac{x+i}{x-i}$ and $d\phi = \frac{-2}{x^2+1}dx$. The prefactors differ by $\sqrt{2\pi}$ from the expected $\sqrt{2}/(z \pm i)$ because of the normalisation.) These are the restrictions to the boundary of the corresponding linear maps $U_\Gamma : \mathrm{Hol}(\Delta_+) \oplus \mathrm{Hol}_0(\Delta_-) \to \mathrm{Hol}(\mathbb{H}_-) \oplus \mathrm{Hol}(\mathbb{H}_+)$, where $\mathrm{Hol}(\mathbb{H}_-)$ (resp. $\mathrm{Hol}(\mathbb{H}_+)$) are the holomorphic functions on the lower (upper) half plane that extend to L^2 functions on $\mathbb{R}$. (For $g_- \in \mathrm{Hol}(\Delta_-)$, U_Γ is holomorphic at i precisely when $g_-(\infty) = 0$, cf. prop. 5.)

On $L^2(\mathbb{R}, dx)$, the complex conjugation $jg(e^{i\phi}) = e^{-i\phi}\overline{g}(e^{i\phi})$, the left-right flips $s_\pm g(e^{i\phi}) = \pm ie^{-i\phi}g(-e^{-i\phi})$ and the rotation $r_\alpha g(e^{i\phi}) = e^{i\alpha}g(e^{i(\phi+2\alpha)})$ (which covers the rotation over 2α on S^1) are given by

$$U_\Gamma j U_\Gamma^{-1} f(x) = i\,\overline{f}(x), \qquad U_\Gamma s_\pm U_\Gamma^{-1} f(x) = \pm g(-x)\,,$$

$$U_\Gamma r_\alpha U_\Gamma^{-1} f(x) = \frac{1}{\sin(\alpha)x + \cos(\alpha)}\, f\left(\frac{\cos(\alpha)x - \sin(\alpha)}{\sin(\alpha)x + \cos(\alpha)}\right)\,.$$

2.3.3 Hilbert transform

For $f \in L^2(\mathbb{R}, dx)$, we define the Cauchy transform

$$\hat{f}(z) := \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{f(x)}{x-z}dx\,.$$

Because $\frac{1}{x-z}$ is uniformly bounded in x by $|\mathrm{Im}(z)|^{-1}$ for each $z \notin \mathbb{R}$, $\hat{f}$ is holomorphic on $\mathbb{C} - \mathbb{R}$. If $f = U_\Gamma g$ with g a bounded function on P, then $f(x)$ decays as $(1+x^2)^{-1/2}$, so that $\hat{f}$ is bounded outside a neighbourhood of $\mathbb{R}$. In particular, $\hat{f}$ is holomorphic in ∞. One checks that for $f = U_\Gamma g$ with $g \in C^0(S^1)$, we have

$$\lim_{\epsilon \downarrow 0} \hat{f}(x+i\epsilon) - \hat{f}(x-i\epsilon) = f(x)\,.$$

We show that for $f = U_\Gamma g$ with $g \in C^1(S^1)$, the limit with the opposite sign also exists, so that $\lim_{\epsilon\downarrow 0} \hat{f}(x+i\epsilon)$ and $\lim_{\epsilon\downarrow 0} \hat{f}(x-i\epsilon)$ are well defined functions.

$$\begin{aligned}
\lim_{\epsilon\downarrow 0} \hat{f}(x+i\epsilon) + \hat{f}(x-i\epsilon) &= \lim_{\epsilon\to 0} \frac{1}{2\pi i}\int_{-\infty}^{\infty} f(t+x)\frac{2t}{t^2+\epsilon^2}dx \\
&= \lim_{\epsilon\to 0} \frac{-1}{2\pi i}\int_{-\infty}^{\infty} f'(t+x)\log(t^2+\epsilon^2)dt \\
&= \frac{-1}{2\pi i}\int_{-\infty}^{\infty} f'(t+x)\log(t^2)dt\,.
\end{aligned}$$

The integrals are finite because the logarithmic singularity at $x = 0$ does not contribute to the integral, and because f' decreases as $\sim \frac{1}{1+x^2}$ for $x \to \pm\infty$. Now if $g \in C^{n+2}(S^1)$, then $f^{(n)} \sim (1+x^2)^{-(n+1)/2}$, so one can take derivatives in the above equations and see that $\lim_{\epsilon\downarrow 0} \hat{f}(x+i\epsilon)$ is a C^n-function in $L^2(\mathbb{R})$.

Proposition 5. *Let $g \in C^\infty(S^1)$. Then $U_\Gamma P_+ g$ and $U_\Gamma P_- g$ are smooth functions on $\mathbb{R}$ that extend to $v_+ \in \mathrm{Hol}(\mathbb{H}_+)$ and $v_- \in \mathrm{Hol}(\mathbb{H}_-)$ respectively. They are given by*

$$v_+(z) = -\hat{f}(z)|_{\mathbb{H}_-}, \quad v_-(z) = \hat{f}(z)|_{\mathbb{H}_+}$$

and are the unique bounded holomorphic functions on $\mathbb{H}_+$ and $\mathbb{H}_-$ that extend continuously to $\mathbb{R}$, satisfy $f = v_+|_\mathbb{R} + v_-|_\mathbb{R}$ with $f = U_\Gamma g$ and go to zero in $\pm\infty$.

Proof. It is clear from the above that $v_+(z) := -\hat{f}(z)|_{\mathbb{H}_-}$ and $v_-(z) := \hat{f}(z)|_{\mathbb{H}_+}$ are bounded holomorphic functions that extend continuously (even smoothly) to $\mathbb{R}$, go to zero in $\pm\infty$, and satisfy $f = v_+ + v_-|_\mathbb{R}$. Any other pair $\tilde{v}_+, \tilde{v}_-$ with these properties satisfies $(v_+ - \tilde{v}_+)|_\mathbb{R} = (\tilde{v}_- - v_-)|\mathbb{R}$, so that the function defined by the l.h.s. on $\mathbb{H}_-$ and by the r.h.s. on $\mathbb{H}_+$ is continuous, hence holomorphic and $\mathbb{C}$, and thus zero because it is bounded and zero at infinity.

If g is smooth on S^1, then so are P_+g and P_-g. They extend to holomorphic functions g_+ and g_- on the unit disc Δ_+ and its complement Δ_-, which are bounded because they extend continuously to the boundary. Now $U_\Gamma g_+ = \frac{\sqrt{\pi}}{z-i} g_+(-i\frac{z+i}{z-i})$ is holomorphic on $\mathbb{H}_-$ because $\frac{1}{z-i}$ and $\frac{z+i}{z-i}$ are, and goes to zero for $x \to \pm\infty$. The function $U_\Gamma g_-(z) = \frac{\sqrt{\pi}}{z-i} g_-(-i\frac{z+i}{z-i})$ is holomorphic on $\mathbb{H}_+ - \{i\}$ for the same reason, and continuous (thus holomorphic and bounded) $\mathbb{H}_+$ because $g_-(\infty) = 0$. It too goes to zero in ∞. Because such functions are unique, this concludes the proof. □

Remark 2. If $s(e^{i\phi}) = \sum_{\frac{1}{2}+\mathbb{Z}} \alpha_n e^{in\chi}$, then $U_\Gamma g_+(-i) = \frac{1}{2} i\sqrt{\pi}\alpha_1$ is the coefficient of 'our' vacuum' $s(\phi) = e^{i\phi/2}$, which has minimal positive energy 1. If we had antiparticles in our theory (which we don't), then $U_\Gamma g_-(i) = -\frac{1}{2}\sqrt{\pi}\alpha_{-\frac{1}{2}}$ would have been the coefficient of the 'other vacuum' $s\phi = e^{-i\phi/2}$ with maximal negative energy -1.

We show that the polarisation $\mathcal{H}_\mathbb{C} = \mathcal{H}_+ \oplus \mathcal{H}_-$ into function living on I and I_c and the polarisation $\mathcal{H}_\mathbb{C} = V_+ \oplus V_-$ into positive and negative energies are in general position, i.e. $\mathcal{H}_\pm \cap V_\pm = \{0\}$. Suppose that $f \in L^2(S^1, d\phi)$ represents a section in $V_+ \cap \mathcal{H}_+$. Then also $\phi * f \in V_+$, where $\phi \in C^\infty(S^1)$. If ϕ is supported in $[-\epsilon, \epsilon]$ and f in $\pi^{-1}(I)$, then $\psi * f$ is supported in I_ϵ, whch is I thickened by ϵ. But since $\phi * f$ is the boundary of a holomorphic function (there are no negative Fourier components), this means that $\phi * f$ must vanish identically. Since $\phi * f$ can be made arbitrarily close to f in $L^2(S^1)$, we must have $f = 0$. In the same vein, one shows that the other 3 intersections are $\{0\}$.

This shows that we can define an antilinear map $\sigma : V_+ \to V_+$ unambiguously by $\sigma(v_+) = jv_-$ iff $v_+ + v_- \in \mathcal{H}(I)$, and that $T_- = (P_+ - Q_-)^2$ is an injective bounded operator.

Finally, we'll need the following nice property of the Cauchy transform:

Proposition 6. *The Cauchy transform is an intertwiner between the unitary representation of* $\mathrm{PSL}(2,\mathbb{R})$ *on* $L^2(\mathbb{R})$ *and the representation on the direct sum* $\mathrm{Hol}(\mathbb{H}_+) \oplus \mathrm{Hol}(\mathbb{H}_-)$*, both given by the formula* $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $U_{g^{-1}}(f)(z) = \frac{1}{cz+d} f(\frac{az+b}{cz+d})$.

Proof. Implement the change of variables $x = \frac{ay+b}{cy+d}$, $dx = \frac{1}{(cy+d)^2} dy$ in

$$U_{g^{-1}}\hat{f}(z) = \frac{1}{2\pi i}\frac{1}{cz+d}\int_{-\infty}^{\infty} \frac{f(x)}{x - \frac{az+b}{cz+d}} dx$$

to obtain

$$\widehat{U_{g^{-1}} f}(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\frac{1}{cy+d} f(\frac{ay+b}{cy+d})}{y - z} dy\,.$$

□

2.3.4 Diagonalisation of T_+ and T_-

The key to finding the modular operators is diagonalising $T_- = Q_+P_+Q_+ + Q_-P_-Q_-$. This amounts to diagonalising $Q'_\pm P'_- Q'_\pm$, where $Q'_\pm = U_\Gamma Q_\pm U_\Gamma^{-1}$ is the projection on $L^2(\mathbb{R}^\pm)$, and we have seen that $P'_- = U_\Gamma P_- U_\Gamma^{-1}$ on $L^2(\mathbb{R})$ is given by $P'_- f(t) = \lim_{\epsilon\downarrow 0} \hat{f}(t + i\epsilon)$, i.e.

$$P'_- f(t) = \frac{1}{2\pi i}\lim_{\epsilon\downarrow 0}\int_{-\infty}^{\infty} f(x)\frac{1}{x - t - i\epsilon} dx\,.$$

Let $V : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ be the unitary given by $Vf = (f_l, f_r)$ with $f_r(t) = e^{t/2} f(e^t)$ and $f_l(t) = e^{t/2} f(-e^t)$. If $f \in \mathrm{Hol}(\mathbb{H}_+)$, then $z \mapsto e^{z/2} f(e^z)$ is holomorphic on the strip $\mathrm{Im}(z) \in (0, \pi)$, and f_r and if_l are the boundaries in $\mathrm{Im}(z) = 0$ and $\mathrm{Im}(z) = \pi$. If $f \in \mathrm{Hol}(\mathbb{H}_-)$, then $z \mapsto e^{z/2} f(e^z)$ is holomorphic on the strip $\mathrm{Im}(z) \in (-\pi, 0)$, and f_r and $-if_l$ are the boundaries in $\mathrm{Im}(z) = 0$ and $\mathrm{Im}(z) = -\pi$.

The projection Q'_+ goes to VQ'_+V^{-1}, which maps (f_l, f_r) to $(0, f_r)$, and VQ'_-V^{-1} maps (f_l, f_r) to or $(f_l, 0)$. We write

$$VQ'_+V^{-1} = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad VQ'_-V^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{1} \end{pmatrix}$$

in the new basis. We now calculate

$$VP'_-V^{-1} = \begin{pmatrix} P_-^{rr} & P_-^{rl} \\ P_-^{lr} & P_-^{ll} \end{pmatrix} .$$

Using the substitution $x = e^u$ for $x > 0$ and $x = -e^u$ for $s < 0$, and multiplying numerator and denominator by e^{-t}, we calculate

$$\begin{aligned}
P_-^{rr} f_r(t) &= \lim_{\epsilon \downarrow 0} \frac{1}{2\pi i} \int_{-\infty}^{\infty} f_r(u) \left(\frac{e^{(u-t)/2}}{e^{u-t} - 1 - i\epsilon e^{-t}} \right) du \\
P_-^{rl} f_l(t) &= -\lim_{\epsilon \downarrow 0} \frac{1}{2\pi i} \int_{-\infty}^{\infty} f_l(u) \left(\frac{e^{(u-t)/2}}{e^{u-t} + 1 + i\epsilon e^{-t}} \right) du \\
P_-^{lr} f_r(t) &= \lim_{\epsilon \downarrow 0} \frac{1}{2\pi i} \int_{-\infty}^{\infty} f_r(u) \left(\frac{e^{(u-t)/2}}{e^{u-t} + 1 - i\epsilon e^{-t}} \right) du \\
P_-^{ll} f_l(t) &= -\lim_{\epsilon \downarrow 0} \frac{1}{2\pi i} \int_{-\infty}^{\infty} f_l(u) \left(\frac{e^{(u-t)/2}}{e^{u-t} - 1 + i\epsilon e^{-t}} \right) du
\end{aligned}$$

If we define

$$s_{\epsilon'}(u) := \frac{1}{2\pi i} \frac{e^{u/2}}{e^u - 1 + i\epsilon'}, \quad c_{\epsilon'}(u) := \frac{1}{2\pi i} \frac{e^{u/2}}{e^u + 1 + i\epsilon'} ,$$

then $P_-^{ll} f_l(t) = -\lim_{\epsilon \downarrow 0} \langle (s_{\epsilon e^{-t}}, \overline{f_r}(\bullet + t) \rangle$. Since t is a fixed value, this amounts to $P_-^{ll} f_l(t) = -\lim_{\epsilon' \downarrow 0} \langle s_{\epsilon'}, \overline{f_r}(\bullet + t) \rangle$, and one obtains similar expressions for the other 3. Since the Fourier transform of a convolution is the product of their Fourier transforms, we calculate the Fourier transforms of s_ϵ and c_ϵ. (And drop the prime on the ϵ.)

Proposition 7. *For $\epsilon^+ > 0$, $\epsilon^- < 0$ and all ϵ respectively, we have*

$$\mathcal{F}s_{\epsilon^+}(k) = \frac{-1}{\sqrt{2\pi}} \frac{(1 - i\epsilon)^{-ik-1/2} e^{\pi k}}{e^{\pi k} + e^{-\pi k}} \quad , \quad \mathcal{F}s_{\epsilon^-}(k) = \frac{1}{\sqrt{2\pi}} \frac{(1 - i\epsilon)^{-ik-1/2} e^{-\pi k}}{e^{\pi k} + e^{-\pi k}}$$

$$\mathcal{F}c_\epsilon(k) = \frac{-i}{\sqrt{2\pi}} \frac{(1 + i\epsilon)^{-ik-1/2}}{e^{\pi k} + e^{-\pi k}} .$$

with $\mathcal{F}$ the Fourier transform $\mathcal{F}f(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x) dx$.

Proof. The function s_ϵ is square integrable and meromorphic, with poles when $e^u = 1 - i\epsilon$, i.e. for $u_m = \frac{1}{2}\log(1+\epsilon^2) - i\arctan(\epsilon) + 2m\pi i$, $m \in \mathbb{Z}$. The residue of $u \mapsto e^{-iku}s_\epsilon(u)$ at the pole u_m is $\frac{1}{2\pi i}(1-i\epsilon)^{-(ik+1/2)}(-e^{2\pi k})^m$, where $(1-i\epsilon)^{-(ik+1/2)}$ is shorthand for $e^{-(ik+1/2)u_0}$.

If $k > 0$, then we complete $[-R, R]$ to a contour along the lower semicircle $\phi \mapsto R\cos(\phi) - iR\sin(\phi)$, $\phi \in [0, \pi]$. On semicircles with radius $R = (2m+1)\pi$, the function s_ϵ is uniformly bounded, whereas e^{-iku} decreases as $e^{-Rk\sin(\phi)}$. The integral along the lower semicircle vanishes, and $\mathcal{F}s_\epsilon(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-iku}s_\epsilon(u)du$ is equal to minus (because the contour is clockwise) $2\pi i$ times the sum of residues in the lower half plane. These are the residues at $m \leq 0$ if $\epsilon > 0$, and at $m < 0$ if $\epsilon > 0$. This is a geometric series, yielding the above formula for $\mathcal{F}s_\epsilon$. For $k > 0$, the contour closes clockwise along the upper half plane, yielding the same result.

The other function c_ϵ is done in an analogous fashion, the poles being $u_m = \frac{1}{2}\log(1+\epsilon^2) + i\arctan(\epsilon) + (2m+1)\pi i$. □

Because the Fourier transfer takes convolution into pointwise multiplication, we have $\mathcal{F}P_-^{rr}f_r(k) = \sqrt{2\pi}\mathcal{F}(f_r)(k)\lim_{\epsilon^-\uparrow 0}\mathcal{F}(s_\epsilon)(-k)$, and similar expressions for the others. (We've exchanged a limit and an integral to obtain this, which is allowed because for bounded f_r, the limit $\lim_{\epsilon\uparrow 0}\langle s_\epsilon, \overline{f}_r(\bullet + t)\rangle$ is uniform in t. To see this, one splits the integral into a part from $t - \sqrt{\epsilon}$ to $t + \sqrt{\epsilon}$ and the remainder.) Thus, with the convention $\widehat{X} := (\mathcal{F} \oplus \mathcal{F}VU_\Gamma)X(\mathcal{F} \oplus \mathcal{F}VU_\Gamma)^{-1}$, we have:

$$\widehat{P}_- = \begin{pmatrix} \frac{e^{\pi k}}{e^{\pi k}+e^{-\pi k}} & \frac{i}{e^{\pi k}+e^{-\pi k}} \\ \frac{-i}{e^{\pi k}+e^{-\pi k}} & \frac{e^{-\pi k}}{e^{\pi k}+e^{-\pi k}} \end{pmatrix}, \quad \widehat{P}_+ = \begin{pmatrix} \frac{e^{-\pi k}}{e^{\pi k}+e^{-\pi k}} & \frac{-i}{e^{\pi k}+e^{-\pi k}} \\ \frac{i}{e^{\pi k}+e^{-\pi k}} & \frac{e^{\pi k}}{e^{\pi k}+e^{-\pi k}} \end{pmatrix}.$$

We now see that $T_\pm$ is diagonalised in this basis: $\widehat{T}_+ = \frac{e^{k\pi}}{e^{k\pi}+e^{-k\pi}}\mathbf{1}$ and $\widehat{T}_- = \frac{e^{-k\pi}}{e^{k\pi}+e^{-k\pi}}\mathbf{1}$. The formulæ in proposition 4 then easily yield:

$$\widehat{j\sigma} = \begin{pmatrix} 1 & -i(e^{k\pi} - e^{-k\pi}) \\ 0 & -1 \end{pmatrix}$$

$$\widehat{ju} = \begin{pmatrix} \frac{2}{e^{k\pi}+e^{-k\pi}} & -i\frac{e^{k\pi}-e^{-k\pi}}{e^{k\pi}+e^{-k\pi}} \\ i\frac{e^{k\pi}-e^{-k\pi}}{e^{k\pi}+e^{-k\pi}} & \frac{-2}{e^{k\pi}+e^{-k\pi}} \end{pmatrix} \quad \widehat{\delta^{\frac{1}{2}}} = \begin{pmatrix} \frac{2}{e^{k\pi}+e^{-k\pi}} & -i\frac{e^{k\pi}-e^{-k\pi}}{e^{k\pi}+e^{-k\pi}} \\ i\frac{e^{k\pi}-e^{-k\pi}}{e^{k\pi}+e^{-k\pi}} & \frac{e^{2k\pi}+e^{-2k\pi}}{e^{k\pi}+e^{-k\pi}} \end{pmatrix}.$$

Since $s'_\pm f(x) = \pm f(-x)$, the two left-right reflections take the shape

$$\widehat{s}_+ = \pm\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

One then checks, by simply multiplying 2×2-matrices, that $u = -ijs_+(P_+ - P_-)$. Combining this with proposition 4, we obtain

Lemma 8 (Tomita-Takesaki Involution)**.** *If I is the right semicircle, then the antilinear Tomita-Takesaki involution $J : \bigwedge V_+ \to \bigwedge V_+$ is given by*

$$J = \kappa^{-1}\Lambda(js_+).$$

The V_+-component $P_+(js_+)P_+$ of this operator maps the boundary of $g \in \mathrm{Hol}(\Delta_+)$ to the boundary of $z \mapsto -i\overline{g}(-\overline{z})$, also holomorphic on Δ_+, which is $-i$ times its Schwarz reflection in the imaginary axis.

The modular evolution Δ^{it} is also readily seen. We only need the action of $\delta^{1/2}$ on V_+, which is given by $\widehat{P}_+\delta^{1/2}\widehat{P}_+ = e^{k\pi}\widehat{P}_+$. Thus $\widehat{P}_+\delta^{i\tau}\widehat{P}_+ = e^{2\pi k i\tau}\widehat{P_+}$, so that $\mathcal{F}^{-1}\delta^{i\tau}f(t) = \mathcal{F}^{-1}f(t+2\pi\tau)$ on V_+. Considered as a holomorphic function on the strip $\mathrm{Im}(z) \in (0, -i\pi)$, this is just a shift to the right. Transforming back along V, we see that $\delta^{i\tau}$ is the unitary dilation by $e^{2\pi\tau}$ on $\mathrm{Hol}(\mathbb{H}_-)$, i.e. $f \mapsto e^{\pi\tau}f(e^{2\pi\tau}z)$. Finally, transforming this by U_Γ, we obtain the unitary induced by the modular flow from $-i$ to i.

Proposition 9 (Modular Flow)**.** *The modular evolution on $\mathcal{F} = \bigwedge V_+$ is given by $\Delta^{i\tau} = \Lambda(\delta^{i\tau})$, where the restriction to V_+ of $\delta^{i\tau}$ is*

$$\delta^{i\tau}g(z) = \frac{1}{-i\sinh(\pi\tau)z + \cosh(\pi\tau)}\, g\left(\frac{\cosh(\pi\tau)z + i\sinh(\pi\tau)}{-i\sinh(\pi\tau)z + \cosh(\pi\tau)}\right) .$$

g is a holomorphic function on the unit disc, the boundary of which represents the section $\tilde{S}^1 \to \mathbb{C}$ $s(\chi) = e^{i\chi}g(e^{2i\chi})$.

In a sense, the nicest way to present the modular operators is on $\mathrm{Hol}(\mathrm{H}_-)$, where the antilinear involution is reflection in the real axis followed by complex conjugation, and the modular flow is the dilation subgroup of $\mathrm{SL}(2,\mathbb{R})$.

We check the behaviour of the modular operators if we rotate over an angle α. Choose $I = [-\pi/2+\alpha, \pi/2+\alpha]$ with $\alpha \neq 0$. Since $r_{\alpha/2}$ covers the rotation over α, we have $Q_+^\alpha = r_{\alpha/2}Q_+^0 r_{-\alpha/2}$. One checks that $r_{\alpha/2}$ commutes with P_+ and j, so that the formulæ in proposition 4 yield $u_\alpha = r_{\alpha/2}u_0 r_{-\alpha/2}$ and $\delta_\alpha = r_{\alpha/2}\delta_0 r_{-\alpha/2}$. So $u_\alpha = -ijs_\alpha$ with $s_\alpha g(e^{i\phi}) = ie^{-i(\phi+\alpha)}g(-e^{-i(\phi+2\alpha)})$. In particular, the modular involution for the complement of a semicircle gets a minus sign.

Corollary 10 (up,down,left,right)**.** *The modular involutions for the left and right semicircle $S^1_l = \{e^{i\phi}\,;\, \phi \in [\pi/2, 3\pi/2]\}$ and $S^1_r = \{e^{i\phi}\,;\, \phi \in [-\pi/2, \pi/2]\}$ are $\kappa^{-1}\Lambda(S)$ and $\kappa^{-1}\Lambda(-S)$, respectively, where $S = -js_0$,*

$$Sg(e^{i\phi}) = i\overline{g}(-e^{-i\phi}) .$$

The modular involutions for the upper semicircle $S^1_+ = \{e^{i\phi}\,;\, \phi \in [0,\pi]\}$ and the lower semicircle $S^1_- = \{e^{i\phi}\,;\, \phi \in [-\pi, 0]\}$ are $\kappa^{-1}\Lambda(F)$ and $\kappa^{-1}\Lambda(-F)$, respectively, where $F = js_{\pi/2}$,

$$Fg(e^{i\phi}) = \overline{g}(e^{-i\phi}) .$$

The Tomita-Takesaki involution $J = \kappa^{-1}\Lambda(F)$ implements an isomorphism between the abstract $\mathcal{A}(S^1_+)$-bimodule $L^2(\mathcal{A}(S^1_+))$, the GNS-representation w.r.t. the vacuum, and the concrete Fock space $\mathcal{F}$.

Proposition 11. *Let $J = \kappa^{-1}\Lambda(F)$. Then $L^2(\mathcal{A}(S^1_+)) \to \mathcal{F} : [a] \mapsto a\Omega$ is an isomorphism of Hilbert $\mathcal{A}(S^1_+)$-bimodules. The left and right action of $\mathcal{A}(S^1_+)$ on $\mathcal{F}$ are given by $a \cdot \xi = a\xi$ and $\xi \cdot a = Ja^*J\xi$, and the positive cone P by $P = \{a \cdot \Omega \cdot a^* \,;\, a \in \mathcal{A}(S^1_+)\}$. The left and right action on $L^2(\mathcal{A}(S^1_+))$ are given by $a \cdot [\xi] = [a\xi]$ and $[\xi] \cdot a = [\xi\Delta^{1/2}a\Delta^{-1/2}]$ (cf. prop. 1), where Δ is the unique modular operator on $L^2(\mathcal{A}(S^1_+))$.*

Proof. Because Ω is cyclic and separating for $\mathcal{A}(S^1_+)$, the map $[a] \mapsto a\Omega$ yields an isomorphism $L^2(\mathcal{A}(S^1_+)) \to \mathcal{F}$ between $\mathcal{F}$ and the GNS-representation of $\mathcal{A}(S^1_+)$ for the ground state. This means that the left action is automatically respected. Because Ja^*J is in $\mathcal{A}(S^1_+)'$ and $J^2 = \mathbf{1}$, the right action $\xi \cdot a = Ja^*J\xi$ commutes with the left action on $\mathcal{F}$. The right actions are intertwined because

$$Ja^*Jb\Omega = S\Delta^{-1/2}a^*\Delta^{1/2}Sb\Omega = S\Delta^{-1/2}a^*\Delta^{1/2}b^*\Omega = b\Delta^{1/2}a\Delta^{-1/2}\,.$$

This shows that the Hilbert bimodules are isomorphic. □

3 The Path Group

Define the path group $PO(V)$ as the group of continuous, piecewise C^1 functions $g : \mathbb{R} \to O(V)$ such that $g^{-1}g'(\phi + 2\pi) - g^{-1}g'(\phi)$ in $\mathrm{Lie}(O(V))$ is independent of ϕ. If the group homomorphisms $s, t : PO(V) \to O(V)$ are the evaluations at 0 and 2π respectively, then the loop group $\Omega O(V) < PO(V)$ is the group of functions g with $s(g) = t(g)$.

An element $g \in PO(V)$ defines an orthogonal transformation of the real Hilbert space $L^2(\mathrm{Moe}^V)$ by left multiplication, and thus an automorphism α_g of the Clifford algebra $\mathrm{Cl}(L^2(\mathrm{Moe}^{V,\mathbb{C}}))$. The same goes for $\mathrm{Cl}(L^2(\mathrm{Moe}^{V,\mathbb{C}}|_I))$, and the automorphisms are compatible with the inclusions.

Remark 3. More generally, if $P \to S^1$ is a principal G-bundle and V a G-representation, then $\Gamma(\mathrm{ad}(P))$ acts on $\Gamma(P \times_G V)$. For Moe and V, the structure group $\mathbb{Z}/2\mathbb{Z}$ acts by ± 1 on V, which twists the fermions but not the loop group.

Twisted loop groups may pop up when S^1 is an orientation-flipping loop in a non-orientable d-dimensional Riemannian manifold M, P is the restriction of the orthogonal frame bundle to S^1, and the vector bundle is the restriction of the tangent bundle to M. For $d = 2n + 1$, the situation is similar to the one we have here: the monodromy (up to $\mathrm{SO}(d)$) yields the ± 1 action, and thus the untwisted loop algebra $\widehat{so}(2n + 1)$. For $d = 2n$, the monodromy (up to $\mathrm{SO}(d)$) yields conjugation by a reflection, and thus the twisted loop algebra corresponding to the unique diagram automorphism of $D_n = \mathfrak{so}(2n)$.

3.1 Discontinuous loops

We show that the von Neumann algebra $\mathcal{A}(S^1)$, which is the strong closure of the C*-algebra $\mathrm{CAR}(S^1)$ w.r.t. the topology induced by the vacuum state $\phi_0(a) := \langle \Omega, \pi_{P_+}(a)\Omega \rangle$, is sensitive enough to know the difference between paths and loops.

Proposition 12. *Every piecewise C^1 path $g \in PO(V)$ defines an automorphism A_g of the C^*-algebra $\mathrm{CAR}(S^1)$. The following are equivalent:*

- *The shifted vacuum $A_g^*\phi_0$ is a normal state.*
- *The automorphism A_g is induced by a unitary operator U_g on $\mathcal{F}$, $A_g(x) = U_g x U_g^{-1}$ for all $x \in \mathrm{CAR}(S^1)$, and thus induces an automorphism of the von Neumann algebra $\mathcal{A}(S^1)$.*
- *The path is a closed loop, $g \in \Omega O(V)$.*

Proof. Because multiplication by g is unitary on $L^2(\mathrm{Moe}^{V,\mathbb{C}})$ and the Clifford algebra norm depends only on the inner product, g extends to an automorphism A_g of the CAR-algebra $\mathrm{CAR}(S^1)$ (the norm closure of the Clifford algebra).

According to Segal's quantisation criterion (p. 480 of [Was98]), the GNS representations $\mathcal{F}$ for ϕ_0 and $\mathcal{F}'$ for $A_g^*\phi_0$ are unitarily equivalent if and only if $[g, P_+]$ is Hilbert-Schmidt. Since both states are pure, and since the normal pure states on $\mathcal{A}(S^1) \simeq B(\mathcal{F})$ are precisely the vector states in $\mathcal{F}$, this is equivalent to the pure state $\alpha_g^*\phi_0$ being normal.

Also, according to p. 480 of [Was98], it follows from this that A_g is induced by a (projective) unitary on Fock space, $A_g = U_g \bullet U_g^{-1}$, if and only if $[g, P_+]$ is Hilbert-Schmidt.

To complete the proof, then, we need only show that multiplication by g is Hilbert-Schmidt if and only if g does not jump. We identify g with its multiplication operator $g \in U_{\mathrm{res}}(\mathcal{H})$, write $\mathcal{H}$ for $L^2(\mathrm{Moe}^{V,\mathbb{C}})$ and P_+ for the positive energy projection, and define the restricted unitary group

$$U_{\mathrm{res}}(\mathcal{H}) := \{u \in U(\mathcal{H})\,;\, \mathrm{Tr}(|[u, P_+]|^2) < \infty\}\,.$$

Pick coordinates on V and write $g = \sum_{i,j=1}^d g_{ij}(\phi) E_{ij}$. Identifying $\mathcal{H}$ with $l^2_{\mathbb{C}}(\mathbb{Z} + \frac{1}{2})$, multiplication by g_{ij} goes to convolution of the fourier coefficients. Now E_{ij} commutes with P_+, so we have

$$\left(\widehat{g_{ij} P_+ \psi_j} - \widehat{P_+ g_{ij} \psi_j}\right)(n) = \sum_{m \in \mathbb{Z} + \frac{1}{2}} (\delta_{m>0} - \delta_{n>0}) \hat{g}_{ij}(n-m) \hat{\psi}_j(m)\,.$$

Since the E_{ij}'s can be taken out, we have

$$\mathrm{Tr}\Big|[\sum_{ij} g_{ij} E_{ij}, P_+]\Big|^2 = \sum_{ij} \mathrm{Tr}\left([g_{ij}, P_+][P_+, g_{ij}^*]\right)\,.$$

Now since $\langle z^n, [g_{ij}, P_+] z^m\rangle = (\delta_{m>0} - \delta_{n>0}) \hat{g}_{ij}(n-m)$, and g_{ij} is real, the above trace reads

$$\begin{aligned}
\mathrm{Tr}\Big|[\sum_{ij} g_{ij} E_{ij}, P_+]\Big|^2 &= \sum_{ij} \sum_{m,n \in \mathbb{Z} + \frac{1}{2}} \hat{g}_{ij}(n-m) \hat{g}_{ij}(m-n) (\delta_{m>0} - \delta_{n>0})^2 \\
&= 2 \sum_{ij} \sum_{m,n \in \mathbb{Z}^{\geq 0} + \frac{1}{2}} \hat{g}_{ij}(n+m) \hat{g}_{ij}(-(m+n)) \\
&= \sum_{ij} \sum_{k \in \mathbb{Z}} |k|\, |\hat{g}_{ij}(k)|^2\,.
\end{aligned}$$

If $g \in \Omega O(V)$ is C^1, then $\sum_{k\in\mathbb{Z}} |k|^2 \hat{g}(k) = \langle g', g' \rangle < \infty$, so the unitary defined by left multiplication on $\mathcal{H}$ certainly lies in $U_{\text{res}}(\mathcal{H})$.

Remark 4. At this point, one can even make due with Sobolev-1/2 functions. We restricted to piecewise C^1 for a good reason though: there are discontinuous Sobolev-1/2 functions that do indeed lift to unitaries! See [PS86], ch. 6.

The continuous 2π-periodic function $\phi \mapsto |\phi|$ on $[-\pi, \pi]$ has Fourier decomposition $k \mapsto (2(-1)^k - 2)\frac{1}{k^2}$, so continuous piecewise C^1 functions also define unitaries in $U_{\text{res}}(\mathcal{H})$. (Continuous piecewise C^1 functions are sums of C^1-functions and multiples of translations of $\phi \mapsto |\phi|$.)

On the other hand, functions which are not 2π-periodic never fulfill these requirements. The function $\phi \mapsto \phi - \pi$ on $[-\pi, \pi]$ has Fourier decomposition $k \mapsto \frac{2\pi(-1)^k i}{k}$. Now if g is not a loop, i.e. if $g(2\pi)g(0)^{-1} = \exp(2\pi X) \in SO(V)$ with $X \neq 0$, then g can be written as $g(\phi) = \exp(2\pi\phi X)\tilde{g}(\phi)$, with $\tilde{g}(\phi)$ a piecewise C^1 loop. If g would lift to $PU(\mathcal{F})$, then so would $\exp(2\pi\phi X)$, which cannot be because the sequence $\sum_{k\in\mathbb{Z}} \frac{1}{|k|}$ diverges. □

Intuitively, this says that even though the C^*-algebra $\text{CAR}(S^1)$ can not tell the difference between piecewise C^1 functions with and without gap (as far as it's concerned, they both yield automorphisms), the quantum probability space that consists of the algebra $\text{CAR}(S^1)$ *equipped with the ground state* $\phi_0(a) := \langle \Omega, \pi_{P_+}(a)\Omega \rangle$ can: if α_g is the automorphism induced by $g \in PO(V)$, then $\alpha_g^* \phi_0$ is normal w.r.t. ϕ_0 if and only if g does not jump.

3.2 Representation of $\Omega O(V)$ and $\Omega\text{Spin}(V)$

We denote by $\Omega_e O(V) \lhd \Omega O(V)$ the normal subgroup of based loops, i.e. loops $g : S^1 \to O(V)$ such that $g(1) = 1$. Then $\Omega O(V) = \Omega_e O(V) \ltimes O(V)$, where the semidirect product is by the adjoint action of $O(V)$. If $\text{Spin}(V)$ is the 2-fold cover of $SO(V)$ (also for $\dim(V) = 2$), then based loops g in $O(V)$ lift to based loops in $\text{Spin}(V)$ if and only if $\pi_1(g)$ is even, which for $\dim(V) = 2$ amounts to g being contractible. We can thus consider $\Omega_e\text{Spin}(V)$ as a normal subgroup of order 2 (equal to $\Omega_e O(V)^0$ for $\dim(V) > 2$), and write $\Omega_e O(V) = \Omega_e SO(V) = \Omega_e\text{Spin}(V) \rtimes \mathbb{Z}/2\mathbb{Z}$.

Proposition 13. *Let $g \mapsto [U_g]$ be the projective unitary representation of $\Omega O(V)$ on $\mathcal{F}$ defined above, and let $d := \dim(V)$. Then the action of U_g on $\mathcal{F} = \mathcal{F}_0 \oplus \mathcal{F}_1$ respects the $\mathbb{Z}/2\mathbb{Z}$-grading if $[g] \in \pi_1(O(V))$ is even and flips it if it is odd; grading is preserved if and only if $g \in \Omega_e\text{Spin}(V) \rtimes O(V) < \Omega O(V)$.*

The corresponding projective representation of the Lie algebra $\Omega\mathfrak{so}(d)$ is a linear unitary representation of the Heisenberg algebra for $d = 2$. For $d = 3$ and $d \geq 5$, it is a unitary representation of the affine Kac-Moody algebra $\widehat{\mathfrak{so}}(d)$ with central charge $\frac{1}{2(d-2)}$. For $d = 4$, it is a unitary representation of $\widehat{\mathfrak{so}}(3) \oplus_D \widehat{\mathfrak{so}}(3)$ with central charge $\frac{1}{4} \oplus \frac{1}{4}$.

Proof. Choose an orthogonal basis $\{\psi^a\, ;\, a = 1, \ldots, d\}$ of V. From this, we derive the orthogonal $\mathbb{R}$-basis of the real Hilbert space $\mathcal{H}_\mathbb{R} := L^2(\text{Moe} \otimes_\mathbb{R} V)$

consisting of the sections

$$c^a(n) := \sqrt{2}\cos(n\phi) \otimes \psi^a \quad \text{and} \quad s^a(n) := \sqrt{2}\sin(n\phi) \otimes \psi^a$$

for $n > 0$, $n \in \mathbb{Z} + \frac{1}{2}$, $a = 1, \ldots, d$.

The complexification $\mathcal{H}_\mathbb{C} := \mathcal{H}_\mathbb{R} \otimes_\mathbb{R} \mathbb{C}$ inherits from $\mathcal{H}_\mathbb{R}$ the unique hermitean and bilinear extensions of the real inner product on $\mathcal{H}_\mathbb{R} \subset \mathcal{H}_\mathbb{C}$, which we denote by $\langle \,\cdot\,, \,\cdot\, \rangle$ and $B(\,\cdot\,, \,\cdot\,)$ respectively. The elements $c^a(n)$ and $s^a(n)$ of the Clifford algebra $\mathrm{Cl}(\mathcal{H}_\mathbb{C}, B)$ satisfy the relations

$$\{c^a(n), c^b(m)\} = \delta_{ab}\delta_{nm}\,, \{s^a(n), s^b(m)\} = \delta_{ab}\delta_{nm} \text{ and } \{c^a(n), s^b(m)\} = 0\,.$$

The operators $\psi^{a*}(n) := \frac{1}{\sqrt{2}}\left(c^a(n) + is^a(n)\right)$ and $\psi^a(n) = \frac{1}{\sqrt{2}}\left(c^a(n) - is^a(n)\right)$, corresponding, recall that $n > 0$, to the complex sections $\phi \mapsto \exp(in\phi)$ and $\phi \mapsto \exp(-in\phi)$ respectively, therefore satisfy

$$\{\psi^a(n), \psi^b(m)\} = 0, \quad \{\psi^a(n), \psi^{b*}(m)\} = 1, \quad \{\psi^{a*}(n), \psi^{b*}(m)\} = 0\,.$$

We now remember that $\mathcal{H}_\mathbb{C}$ is an S^1-representation with generator D, and as such splits into positive and negative energy parts. We write $\mathcal{H}_\mathbb{C} = V_+ \oplus V_-$, where V_+ and V_- are the images of the positive and negative spectral projections P_+ and P_- of D. The $\psi^{a*}(n)$ with $n > 0$ form a basis for V_+, whereas the $\psi^a(n)$ with $n > 0$ constitute a basis of V_-. The Fock space $\mathcal{F}^0 = \mathrm{Cl}(\mathcal{H}_\mathbb{C})/(\mathrm{Cl}(\mathcal{H}_\mathbb{C})V_-)$ is a quotient of Clifford representations, and as such a Clifford representation itself. Since $B(V_+, V_+) = 0$ and $B(V_-, V_-) = 0$, the inclusion $\bigwedge V_+ = \mathrm{Cl}(V_+) \hookrightarrow \mathrm{Cl}(\mathcal{H}_\mathbb{C})$ yields an isomorphism $\bigwedge V_+ \simeq \mathcal{F}^0$. Since V_+ is a Hilbert space, this endows $\mathcal{F}^0$ with a Hermitean form, and the Clifford representation extends to its closure, the Hilbert space $\mathcal{F}$. It is a $*$-representation w.r.t. the involution $v_1 \ldots v_n \mapsto \overline{v}_1 \ldots \overline{v}_n$ of $\mathrm{Cl}(\mathcal{H}_\mathbb{C})$.

The projection $P_+ : \mathcal{H}_\mathbb{C} \to V_+$ is bijective when restricted to $\mathcal{H}_\mathbb{R}$, and endows $\mathcal{H}_\mathbb{R}$ with the complex structure $J = i\mathrm{sg}(D)$. The hermitean form (w.r.t. J) is the unique one that agrees with the real inner product on the 'symmetric' sections, i.e. the ones for which $s(-\phi)$ agrees with $s(\phi)$ after parallel transport. We could use this to identify $\mathcal{F}$ with $\bigwedge_J L^2(\mathrm{Moe} \otimes V)$, but we will not make use of this. Instead, we use the isomorphism of $\mathcal{F}$ with $\bigwedge V_+$ to give an explicit basis for $\mathcal{F}$;

$$\psi^{1*}(n^1_{k_1}) \ldots \psi^{1*}(n^1_1) \ldots \psi^{d*}(n^d_{k_d}) \ldots \psi^{d*}(n^d_1)\,\Omega$$

with $n^a_{k_a} > \ldots > n^a_1$ for all $a = 1, \ldots, d$.

An orthogonal transformation $u \in O(\mathcal{H}_\mathbb{R})$ induces an automorphism α_u of $\mathrm{Cl}(\mathcal{H}_\mathbb{C})$. If there exists a $U \in U(\mathcal{F})$ so that $\alpha_u(A)v = UAU^{-1}v$ for all $v \in \mathcal{F}$, then U (which is determined up to S^1) is a second quantisation of u. For example, if the complexification of u happens to respects V_+, then the induced unitary on $\mathcal{F} \simeq \bigwedge V_+$ is called the *canonical* quantisation of u. In particular, since the action of the constant loops commutes with rotation, the $O(V)$ action is canonically quantised.

The Lie algebra $\mathrm{Cl}^{\leq 2}_+(\mathcal{H}_\mathbb{C})$ of even Clifford elements of degree ≤ 2 acts by commutation on $\mathcal{H}_\mathbb{C} \subset \mathrm{Cl}(\mathcal{H}_\mathbb{C})$. For instance, $v \cdot w$ acts as $X \mapsto [v \cdot w, X] =$

$vB(w,X) - wB(v,X)$. Introduce the notation $\psi^{a*}(-n) := \psi^a(n)$, so that $\psi^{a*}(n)$ is the Clifford element corresponding to $e^{in\phi} \otimes \psi^a$ for all $n \in \mathbb{Z} + \frac{1}{2}$. Note that for $a \neq b$ and $k \in \mathbb{Z}$, the expression

$$E^{ab}(k) := \sum_{n \in \mathbb{Z}} \psi^a(k-n)\psi^b(n)$$

is well defined on $\mathcal{F}^0$, and that $E^{ab}(k)\Omega = 0$ for $k \geq 0$. Using the Clifford relations and the fact that $E^{ab}(k)v$ only involves finitely many terms if v is one of the basis elements of $\mathcal{F}$, one sees that $[E^{ab}(k), \psi^a(m)] = -\psi^b(m+k)$ and $[E^{ab}(k), \psi^b(m)] = \psi^a(m+k)$, so that commuting with the densely defined, unbounded operator $E^{ab}(k)$ on $\mathcal{F}$ corresponds to the action of $e^{-ik\phi}(e_{ab} - e_{ba}) \in L\mathfrak{so}(V) \otimes \mathbb{C}$. Because the Clifford algebra is generated by $\mathcal{H}_{\mathbb{C}}$, and because the action (by commutation) of $c(E^{ab}(k), E^{a'b'}(k')) := [E^{ab}(k), E^{a'b'}(k')] - \delta_{ba'}E^{ab'}(k+k') - \delta_{b'a}E^{a'b}(k+k')$ on $\mathcal{H}_{\mathbb{C}}$ is trivial, $c(E^{ab}(k), E^{a'b'}(k'))$ must be an element of the commutant of $\mathrm{Cl}(\mathcal{H}_{\mathbb{C}})$ on $\mathcal{F}$, which is $\mathbb{C}\mathbf{1}$. In other words, c is a cocycle, and its values are determined by evaluating in the ground state Ω.

Using $E^{ab}(k)^\dagger = -E^{ab}(-k)^\dagger$, the Clifford relations, and the fact that $\psi^a(n)\Omega = 0$ for $n > 0$, and the fact that the vectors $E^{ab}(k)\Omega$ are eigenstates of $D = \frac{2}{i}\frac{d}{d\phi}$ with energy $2k$ and therefore orthogonal unless k agrees, one calculates

$$\begin{aligned} c(E^{ab}(k), E^{a'b'}(k')) &= \langle \Omega, [E^{ab}(k), E^{a'b'}(k')]\Omega \rangle \\ &= -\langle E^{ab}(-k)\Omega, E^{a'b'}(k')\Omega \rangle + \langle E^{a'b'}(-k')\Omega, E^{ab}(k)\Omega \rangle \\ &= -k\delta_{k+k',0}(\delta_{aa'}\delta_{bb'} - \delta_{ab'}\delta_{a'b}) \\ &= k\delta_{k+k',0}\frac{1}{2(d-2)}\kappa(e_{ab} - e_{ba}, e_{a'b'} - e_{b'a'})\,. \end{aligned}$$

For $d = 3$ and $d \geq 5$, this shows that we have a representation of central charge $c = \frac{1}{2(d-2)}$ for the affine Kac-Moody algebra $\widehat{\mathfrak{so}}(d)$. For $d = 4$, the special orthogonal Lie algebra $\mathfrak{so}(4)$ is not simple, but isomorphic to $\mathfrak{so}(4) \simeq \mathfrak{so}(3) \oplus \mathfrak{so}(3)$. If we denote $E_{ij} = e_{ij} - e_{ji}$, then the two commuting bases of $\mathfrak{so}(3)$ are given by

$$E^+_{12} = \tfrac{1}{2}\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} E^+_{13} = \tfrac{1}{2}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} E^+_{23} = \tfrac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

and

$$E^-_{12} = \tfrac{1}{2}\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} E^-_{13} = \tfrac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} E^-_{23} = \tfrac{1}{2}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

This shows that the Killing form $\kappa(X,Y) = (d-2)\mathrm{tr}(XY)$ on $\mathfrak{so}(3)$ is precisely (with the right scaling) the restriction of the Killing form on $\mathfrak{so}(4)$. Thus we get a single $\widehat{\mathfrak{so}}(3)$ representation of central charge $c = \frac{1}{2}$ for $\dim(V) = 3$, and two commuting $\widehat{\mathfrak{so}}(3)$ representations of central charge $\frac{1}{4}$ for $\dim(V) = 4$.

Since $E^{ab}(k)^\dagger = -E^{ab}(-k)^\dagger$, the operators $\cos(k\phi)(e_{ab}-e_{ba})$ and $\sin(k\phi)(e_{ab}-e_{ba})$ on $\mathcal{H}_\mathbb{R}$ are represented by selfadjoint elements, which can be exponentiated. The projective Lie algebra representation of (the Fourier monomials in) $\mathrm{Lie}(\Omega_e O(V))$ integrates to a projective group representation of $\Omega_e O(V)^0 = \Omega_e \mathrm{Spin}(V)$. Because the generators consist of even elements, the $\mathbb{Z}/2\mathbb{Z}$-grading of $\mathcal{F}$ is preserved by these unitaries. Since we already know that the constant loops $O(V)$ have a canonical lift to $U(\mathcal{F})$, we need only determine what happens to the noncontractible loop $g(\phi) = \cos(\phi)(e_{aa} + e_{bb}) + \sin(\phi)(e_{ba} - e_{ab}) + \sum_{c\neq a,b} e_{cc}$ that generates $\Omega_e O(V)/\Omega_e \mathrm{Spin}(V)$ (which is $\mathbb{Z}$ for $d = 2$ and $\mathbb{Z}/2\mathbb{Z}$ for $d > 2$).

In order to do this, we write $V_{ab} := \mathrm{span}(\psi^a, \psi^b)$, so that $V = V_{ab} \oplus V_{ab}^\perp$, and consider $\mathcal{F}$ as the super tensor product $\mathcal{F}(V) = \mathcal{F}(V_{ab})\widehat{\otimes}\mathcal{F}(V_{ab}^\perp)$ accordingly. Because g acts trivially on $\mathrm{Moe} \otimes_\mathbb{R} V_{ab}^\perp$, we need not worry about the second part. We introduce (for all $n \in \mathbb{Z} + \frac{1}{2}$) the vectors $\omega_\pm(n) := \psi^a(n) \pm i\psi^b(n)$, on which g acts by $g \cdot \omega_\pm(n) = \omega_\pm(n \pm 1)$. Multiplication by g is supposed to shift the energy of each individual fermion by ± 1. We note that $\omega_\pm(n)^* = \omega_\mp(-n)$, and that $\{\omega_+(-n), \omega_-(-m)\,;\, n, m > 0\}$ is a $\mathbb{C}$-basis of V_+^{ab}. We therefore have a basis

$$\omega_-(-n_k) \ldots \omega_-(-n_1)\omega_+(-m_l) \ldots \omega_+(-m_1)\,\Omega$$

of $\mathcal{F}(V_{ab})$, where $n_k > \ldots > n_1 > 0$ and $m_l > \ldots > m_1 > 0$.

Any quantisation U of g must satisfy $U\omega_+(n)U^\dagger = \omega_+(n+1)$ for all $n \in \mathbb{Z}+\frac{1}{2}$ (which implies $U\omega_-(n)U^\dagger = \omega_-(n-1)$), so that its action on basis elements is prescribed by the value of $U\Omega$,

$$\begin{aligned} U\omega_-(-n_k) \ldots \omega_-(-n_1)\omega_+(-m_l) \ldots \omega_+(-m_1)\,\Omega \;&= \\ \omega_-(-n_k - 1) \ldots \omega_-(-n_1 - 1)\omega_+(-m_l + 1) \ldots \omega_+(-m_1 + 1)\,U\Omega\,. \end{aligned}$$

Since $U\omega_+(n)\Omega = 0$ for $n > 0$ implies $\omega_-(-k)^*U\Omega = 0$ for $k > 1$, and similarly $U\omega_-(n)\Omega = 0$ for $n > 0$ implies both $\omega_+(-k)^*U\Omega = 0$ for $k > 0$ and $\omega_-(-\frac{1}{2})U\Omega = 0$, we must have that $U\Omega$ is a unimodular multiple of $\omega_-(-\frac{1}{2})\Omega$. One checks that the unitary operator defined like this, corresponding to a shift by one in the zero point energy, is well defined and respects the commutation relations. Since it either decreases (if the original state contains a $\omega_+(-\frac{1}{2})$-excitation) or increases (if it does not) the number of fermions, this operator reverses the $\mathbb{Z}/2\mathbb{Z}$ grading of $\mathcal{F}$. □

Remark 5. This is the same as in the finite dimensional case: the projective action of $O(V)$ on a $\mathrm{Cliff}(V)$-module is given by even operators for $O(V)^0 = SO(V)$, namely the exponentials of $E_{ij} - E_{ji} = \psi_i\psi_j - \psi_j\psi_i$, and by the odd operators ψ_i for reflections.

4 Defects and Sectors

For $g \in PO(V)$, we define the *Fer(V)-Fer(V)* defect D_g. (For the definition of a defect, see [BDH09].) If $E \to M$ is a bundle of real inner product spaces with fibre V and connection ∇, than for any loop $L : S^1 \to M$, the pull back $L^*E \to S^1$ is isomorphic to either the trivial bundle or to $\mathrm{Moe} \otimes_{\mathbb{R}} V$. In the latter case, the connection ∇ yields an orthogonal isomorphism $\Gamma_{L_2}(L^*E) \to \Gamma_{L_2}(\mathrm{Moe} \otimes_{\mathbb{R}} V)$, which respects the smooth structure precisely when the holonomy is -1. The idea is roughly to define a defect as the conformal net arising in this way, but described in terms of Moe only.

4.1 Definition of defects

First of all, note that for $g \in PO(V)$, the action of g on $L^2(\mathrm{Moe}^{V,\mathbb{C}})$ is local. We consider $L^2(\mathrm{Moe}^{V,\mathbb{C}}) = L^2(\mathrm{Moe}^{V,\mathbb{C}}|_I) \oplus L^2(\mathrm{Moe}^{V,\mathbb{C}}|_{I^c})$ in the obvious fashion ($I^c := S^1 - I$), and split $\mathrm{CAR}(S^1) = \mathrm{CAR}(I) \hat{\otimes} \mathrm{CAR}(I^c)$ accordingly as a super tensorproduct of graded C^* algebras. Then by locality, g induces automorphisms of $\mathrm{CAR}(I)$ and $\mathrm{CAR}(I^c)$ separately, and $\alpha_g(a \hat{\otimes} b) = \alpha_g^I(a) \hat{\otimes} \alpha_g^{I^c}(b)$.

We show that if g is continuous on the interior points of $I \subset \mathbb{R}$, then the automorphism α_g on $\mathrm{CAR}(I)$ is weakly continuous, and induces an automorphism of $\mathcal{A}(I)$.

Find a piecewise C^1 loop $\tilde{g} \in \Omega O(V)^\circ$ such that $\tilde{g}|_I = g|_I$. The automorphisms α_g and $\alpha_{\tilde{g}}$ agree on $\mathrm{CAR}(I)$ because their actions on $L^2(\mathrm{Moe}^{V,\mathbb{C}}|_I)$ agree. The automorphism $\alpha_{\tilde{g}}$ lifts to a (projective) unitary on $\mathcal{F}$, that is to say, there exists a unitary $U_{\tilde{g}}$ so that $\alpha_{\tilde{g}}(A) = U_{\tilde{g}} A U_{\tilde{g}}^{-1}$. It is possible to choose $U_{\tilde{g}} \in \mathcal{A}(J)$ for all $J \supset\supset I$, but usually one cannot have $U_{\tilde{g}} \in \mathcal{A}(I)$.

Another choice $\tilde{g}' = \tilde{g}\delta$ with $\delta|_{I^c} = 1$ yields a different unitary but the same automorphism: $U_\delta a U_\delta^{-1} = a$. The Lie algebra generators for U_δ can be written explicitly in terms of even elements of $\mathrm{CAR}(I^c)$, so that $U_\delta \in \mathcal{A}(I^c)$ graded commutes (and therefore, due to evenness, commutes) with $a \in \mathcal{A}(I)$.

This yields a *weakly continuous* automorphism α_g on the v.N-algebra $\mathcal{A}(I)$, considered as a weakly closed subalgebra of $\mathcal{A}(J)$ for some $J \supset\supset I$. It extends the automorphism of $\mathrm{CAR}(I)$ which we already had, and the fact that $\mathrm{CAR}(I)$ is (weakly) dense in $\mathcal{A}(I)$ is another way to see that α_g does not depend on the choice of $\tilde{g}$. As noted before, the automorphism $\alpha_g \in \mathrm{Aut}(\mathrm{CAR}(I))$ does *not* lift to a weakly continuous automorphism of $\mathcal{A}(I)$ if g has a discontinuity in the *interior* of I.

Definition 2. *Let $g \in PO(V)$ such that g is 1 on a neighbourhood of the lower semicircle S^1_-. We define the* Fer(V)-Fer(V) *defect D_g as follows:*

- *On all intervals (bicoloured or not), $D_g(I) := \mathcal{A}(I)$.*
- *If I and J are both either 1- or bicoloured, then $D_g(j) := \mathcal{A}(j)$. If $j : I \to J$ maps a 1-coloured into a bicoloured interval, then $D_g(j) := \alpha_g \circ \mathcal{A}(j)$.*

- *Part of the structure of a* Fer(V)-Fer(V) *defect is a natural isomorphism between* D_g *restricted to the white (or black) intervals and the net* Fer(V)*. This is just the identity.*

Remark 6. Note that α_g is well defined on the image of $\mathcal{A}(I)$ in $\mathcal{A}(J)$: the 2-coloured interval J comes with a local parameterisation $c : \mathbb{R} \to J$ such that $c(\mathbb{R}^{\geq 0})$ is contained in the white and $c(\mathbb{R}^{\leq 0})$ in the black part of J. By writing $j = i \circ j_0$, where $j_0 : I \to \mathrm{Im}(j)$ is just j with a restricted target and $i : \mathrm{Im}(j) \to j$ is the inclusion, we can assume without loss of generality that j is an inclusion. Say I is white (the black case is proven analogously). Write I as the union of $I_0 := I \cap c(\mathbb{R})$ and $I_+ := I - I \cap c(\mathbb{R}_{\leq 1})$. On I_0, the map c^{-1} composed with stereographic projection $p : \mathbb{R} \to S^1$ realises I_0 as a the right semicircle S^1_r. Extend $p \circ c^{-1}$ (in any way) to a diffeomorphism $\tilde{c}^{-1} : I \to S^1 - S^1_{l,+}$. Then the map $A \mapsto \mathcal{A}(c) \circ \alpha_g \circ \mathcal{A}(c)^{-1}(A)$ is the identity on $\mathcal{A}(I_+)$, because $g|_{S^1_-} = 1$. As $\mathcal{A}(I)$ is generated by $\mathcal{A}(I_0)$ and $\mathcal{A}(I_+)$ by local additivity, the result does not depend on the way in which the parametrisation has been extended.

Remark 7. Note that this is a *continuous* functor Int $\to$ vNalg. Even if $\bigcup_{\mathbb{N}} I_n = I$ and the 'changing point' is in ∂I, then the restriction of $\alpha_g : \mathcal{A}(I) \to \mathcal{A}(I)$ to $\mathcal{A}(I_n)$ is precisely $\alpha_g : \mathcal{A}(I_n) \to \mathcal{A}(I_n)$.

Proposition 14. *The precosheaf* D_g *defined above is in fact a defect.*

We only look at the vacuum axiom (all the other ones are immediate). First, we follow [DH12], and define $\lambda(a)\xi = D_g(a)\xi$, which is a left action of $\mathcal{A}(I)$ on $L^2(\mathcal{A}(S^1_+)) \simeq \mathcal{F}$. Note that $L^2(\mathcal{A}(S^1_+)) \simeq \mathcal{F}$ as an $\mathcal{A}(S^1_+)$-bimodule, where $\mathcal{F}$ is equipped with the left action $a \cdot \xi = a\xi$ and commuting right action $\xi \cdot a = Ja^\dagger J\xi = \kappa^{-1}\Lambda(F)a^\dagger\kappa^{-1}\Lambda(F)\xi$ (we used the explicit formula for the modular operators). Therefore, we simply have $\lambda(a)\xi = a\xi$.

Again following [DH12], we introduce the *left* action of $\mathcal{A}(S^1_-)$ on $L^2(\mathcal{A}(S^1_+)) \simeq \mathcal{F}$ given by $\rho(b)\xi := (-1)^{|b||\xi|}\xi \cdot D_g(F)(b)$. If we unravel the definitions, then we see that $D_g(F) = \mathcal{A}(F)$ (both semicircles are bicoloured), and since $Fg(e^{i\phi}) = \overline{g}(e^{-i\phi})$ is antilinear and orientation reversing, $\mathcal{A}(F)(b) = \Lambda(F)(\#_i b)\Lambda(F)$ is – hopefully – the appropriate linear *-homomorphism $\mathcal{A}(S^1_-) \to \overline{\mathcal{A}(S^1_+)}^{\mathrm{op}}$. The left action ρ of $\mathcal{A}(S^1_-)$ on $\mathcal{F} \simeq L^2(S^1_+)$ thus takes the shape

$$\begin{aligned}
\rho(b)\xi &= (-1)^{|b||\xi|}\xi \cdot \mathcal{A}(F)(b) \\
&= (-1)^{|b||\xi|}\xi \cdot \Lambda(F)\#_i b\Lambda(F) \\
&= (-1)^{|b||\xi|}\kappa^{-1}\Lambda(F)(\Lambda(F)\#_i b\Lambda(F))\kappa^{-1}\Lambda(F)\xi \\
&= (-1)^{|b||\xi|}\kappa^{-1}\#_i b\kappa\xi \\
&= b\xi\,.
\end{aligned}$$

(We used $\kappa^{-1}\Lambda(F) = \Lambda(F)\kappa$, $\Lambda(F)^2 = 1$, and in the last line we checked case by case what happens if b and ξ are odd or even.)

For the trivial defect, the transition functions $\mathcal{A}(J) \to \mathcal{A}(S^1_+)$ and $\mathcal{A}(J') \to \mathcal{A}(S^1_-)$ are just the embeddings, and the action $\lambda\hat{\otimes}\rho : \mathcal{A}(J)\hat{\otimes}\mathcal{A}(J') \to B(\mathcal{F})$

of the $\mathbb{Z}/2\mathbb{Z}$-graded tensorproduct is simply $\lambda\hat{\otimes}\rho(a\hat{\otimes}b)\xi = ab\xi$. This obviously extends to $\mathcal{A}(J \cup J')$.

If the defect is given by a path g, then the inclusions $\mathcal{A}(J) \to \mathcal{A}(S^1_+)$ and $\mathcal{A}(J') \to \mathcal{A}(S^1_-)$ are twisted by α_g, so that $\lambda\hat{\otimes}\rho(a\hat{\otimes}b)\xi = \alpha_g(a)\alpha_g(b)\xi$. Since g does not not have a discontinuity in the interior of $J \cap J' = S^1_r$, α_g extends to an isomorphism of $\mathcal{A}(J \cap J')$, and we can write $\alpha_g(a)\alpha_g(b) = \alpha_g(ab)$. Thus $\lambda\hat{\otimes}\rho$ extends to the twisted action α_g of $\mathcal{A}(S^1_r)$ on $\mathcal{F}$. □

Remark 8. I cheated a bit here, because I really should have shown that the whole procedure in [DH12] for turning the orientation reversing diffeomorphism $e^{i\phi} \to e^{-i\phi}$ into pin diffeomorphism and then into a morphism $\mathcal{A}(S^1_-) \to \overline{\mathcal{A}(S^1_+)}^{\text{op}}$ does indeed yield $b \mapsto \Lambda(F)(\#_i b)\Lambda(F)$. So the proof that the defect is in fact a defect is slightly defect. However, the fact that D_g fulfills the vacuum axiom if and only if $\mathrm{Fer}(V)$ does is independent of the precise form of $\mathcal{A}(z \mapsto \overline{z})$, so the definition of defect is correct also if the above proof is not.

If $\mathcal{A}$ is a conformal net and $f : I \to J$ is an antilinear orientation reversing pin morphism, then $\mathcal{A}(f) : \mathcal{A}(I) \to \mathcal{A}(J)^{op}$ is a morphism (linear or antilinear) of $\mathbb{Z}/2$-graded von Neumann algebras. This is *not* an antilinear antihomomorphism $\mathcal{A}(I) \to \mathcal{A}(J)$ of von Neumann algebras because we defined $a \cdot_{\text{op}} b = (-1)^{|a||b|} b \cdot a$ and $(a)^{op-*} = (-1)^{|a|} a^*$. However, $a \mapsto i^{|a|}\mathcal{A}(f)(a)$ *is* an antilinear antihomomorphism. For $\mathrm{Fer}(V)$, it is given by $a \mapsto \Lambda(f)a^\dagger\Lambda(f^{-1})$.

Definition 3. *For the free fermionic net, an antilinear orientation reversing pin morphism $f : I \to J$ gives rise to the* linear homomorphism $\mathcal{A}(I) \to \overline{\mathcal{A}(J)}^{\text{op}}$: $A \mapsto \Lambda(f)\#_i(A)\Lambda(f^{-1})$ *(with $\Lambda(f)$ the Virasoro operators) and to the* antilinear anti-homomorphism $\mathcal{A}(I) \to \mathcal{A}(J) : A \mapsto \Lambda(f)A^\dagger\Lambda(f^{-1})$.

We'll work with the antilinear anti-involutions as much as possible and restrict the use of $\#_i$ to the bare minimum.

We have defined a separate defect D_g for each $g \in PO(V)$, but it turns out that up to isomorphism, D_g depends only on the class of g in $\Omega O(V)\backslash PO(V) \simeq \Delta O(V)\backslash O(V) \times O(V)$. The element of $\Delta O(V)\backslash O(V) \times O(V)$ corresponding to g is of course just $[t(g), s(g)]$.

Proposition 15. *If $g, h \in PO(V)$, then defects D_g and D_h are isomorphic if and only if $hg^{-1} \in \Omega O(V)$. I.e., there exists an invertible natural transformation $N : D_g \to D_h$ that restricts to the identity on the black and on the white intervals. (This is* not *an isomorphism in the 3-category of conformal nets!)*

Proof. Let I be a 1-coloured and J a bigger 2-coloured interval, so that the inclusion $I \hookrightarrow J$ induces the twist by g on D_g, i.e. $\alpha_g : \mathcal{A}_g(I) \hookrightarrow \mathcal{A}_g(J)$ and similarly the twist by h on D_h, $\alpha_h : \mathcal{A}_h(I) \hookrightarrow \mathcal{A}_h(J)$. (Recall that $D_h(I)$ is simply $\mathcal{A}(I)$. We write subscripts to remember which inclusions to use in the precosheaf structure.)

If D_g and D_h are isomorphic, then there exists an invertible, continuous natural transformation between them. That is, there must exist for each interval

I (be it coloured or not) a von Neumann-algebra isomorphism $N(I) : \mathcal{A}_g(I) \to \mathcal{A}_h(I)$ which is compatible with inclusions and limits of intervals.

Since the natural isomorphism between $\mathrm{Fer}(V)$ and the restriction to black (or white) intervals of D_g is part of the data for the defect, we require $N : D_g \to D_h$ to respect it. This means that for black (or white) intervals I, the isomorphism $N(I) : D_g(I) \to D_h(I)$ is the identity under the natural isomorphisms $D_g(I) \simeq \mathcal{A}(I)$ and $D_h(I) \simeq \mathcal{A}(I)$, which, according to the definition of our defect, are just the equalities $D_g(I) = \mathcal{A}(I)$ and $D_h(I) = \mathcal{A}(I)$. In plain english: $N(I)$ is the identity if I is either black or white.

We exploit the compatibility with inclusions. Let J be a bicoloured interval, and let J_B and J_W be the black and white parts of J. (Both 1-coloured, of course.) The commuting diagram

$$\begin{array}{ccc} D_g(J_W) & \xrightarrow{N(J_W)=\mathrm{Id}} & D_h(J_W) \\ {\scriptstyle\alpha_g}\downarrow & & {\scriptstyle\alpha_h}\downarrow \\ D_g(J) & \xrightarrow{N(J)} & D_h(J) \\ {\scriptstyle\alpha_g}\uparrow & & {\scriptstyle\alpha_h}\uparrow \\ D_g(J_B) & \xrightarrow{N(J_B)=\mathrm{Id}} & D_h(J_B) \end{array}$$

shows that $N(J) = \alpha_h \circ \alpha_g^{-1} = \alpha_{hg^{-1}}$ on the image of $D_g(J_B)$ and on the image of $D_g(J_W)$. By additivity, the images of $D_g(J_B)$ and of $D_g(J_W)$ generate $D_g(J)$, so that the isomorphism (of von Neumann algebras!) $N(J)$ is, under the identification $D_g(J) \simeq \mathcal{A}(J) \simeq D_h(J)$, a weakly continuous extension of the automorphism $\alpha_{hg^{-1}} : \mathrm{CAR}(J) \to \mathrm{CAR}(J)$. According to proposition 12, this implies that gh^{-1} does not have discontinuities in the interior of J, and because J is arbitrary that $gh^{-1} \in \Omega O(V)$.

It is not hard to check that $N(I) = \mathrm{Id}$ for I black or white and $N(J) = \alpha_{hg^{-1}}$ for bicoloured J is indeed an isomorphism $D_g \to D_h$ □

Another definition of defects, equivalent up to isomorphism, is the following one, which uses the holonomy of a connection rather than a constant section. An element $G \in O(V)$ can be considered as a constant loop, and therefore lifts to an element $U_G \in U(\mathcal{F})$. This yields a spin representation on $\mathcal{F}$, and thus a homomorphism $O(V) \to \mathrm{Aut}(\mathcal{A}(S^1))$ given by $\alpha_G(a) = U_G a U_G^{-1}$. This gives rise to a defect D_G.

Definition 4. *Let $G \in O(V)$. The $\mathrm{Fer}(V) - \mathrm{Fer}(V)$ defect D_G is defined as follows:*

- *On all intervals, coloured or not, $D_G(I) := \mathcal{A}(I)$.*
- *If I and J are both white, both black, or if I is black and J is bicoloured, and if $j : I \to J$, then $D_g(j) = \mathcal{A}(j)$. If I is white and J bicoloured, then $D_G(j) = \alpha_G \circ \mathcal{A}(j)$.*

- *The natural transformation between D_G restricted to the black or white intervals and $\mathcal{A}$ is the identity.*

Proposition 16. *Let $G \in O(V)$ and $g \in PO(V)$ with $g|_{S^1_-} = 1$. Then the defects D_G and D_g are isomorphic if and only if $G = g_i^{-1} g_f$.*

Proof. Left multiplication with $(g_L \overline{g}_L)^{-1}$ (for notation, see below) does not alter the isomorphism class of D_g, so we may assume that $g_{S^1_L} = 1$ as well as $g_{S^1_-} = 1$. Suppose that $g_f = G$. Then the function $h : S^1 \to O(V)$ defined by $h(z) = g(z)$ for $\Re(z) \leq 0$ and $h(z) = G^{-1} g(z)$ for $\Re(z) > 0$ is continuous in i, but discontinuous in $-i$. The natural transformation $D_G \to D_g$ is then given by $N(I) = \mathrm{Id}$ on black and white intervals, If J is bicoloured and $J \subset S^1$ with $i \in J$ the 'turning point', $-i \notin J$, and the white and black parts are $J_W = J \cap S^1_L$, $J_B = J \cap S^1_R$, then $N(J)(A) = \alpha_h(A)$. This is well defined because h is continuous on J. If J only has a parameterised collar, then $N(J)$ is defined as in remark 6, with the difference that the right part A_+ maps to $\alpha_G(A_+)$ (which is globally defined) rather than A_+. This shows that D_G is isomorphic to D_g if $g_i = 1$ and $g_f = G$, and by the previous proposition not to any D_q with $q_i^{-1} q_f \neq G$. □

4.2 Definition of sectors

For the definition of a sector, see [BDH09]. We adapt the definition in the sense that we require the Hilbert space to be $\mathbb{Z}/2\mathbb{Z}$-graded, and the homomorphisms ρ_I to be homomorphisms of $\mathbb{Z}/2\mathbb{Z}$-graded von Neumann algebras. A morphism of sectors will be a *grading-preserving* invertible isometry of Hilbert spaces that intertwines the homomorphisms.

Definition 5. *Let g and h be 1 on a neighborhood of S^1_- and have no discontinuities outside i. We define a $D_g - D_h$ sector $\mathcal{F}_{g,h}$ as follows. Define $\overline{h}(e^{i\phi}) = h(e^{-i\phi})$, so that $\overline{h} = 1$ on a neighbourhood of S^1_+, and has no discontinuities outside $-i$.*

- *$\mathcal{F}_{g,h} = \mathcal{F}$ as a Hilbert space.*
- *If $i \in I$, then $\mathcal{A}(I)$ acts by $A : \xi \mapsto \alpha_{\overline{h}}(A)\xi$. If $-i \in I$, then by $A : \xi \mapsto \alpha_g(A)\xi$, and if $\pm i \notin I$, then by $\xi \mapsto \alpha_{g\overline{h}}(A)\xi$.*

This action is compatible with the precosheaf structure of D_g and D_h; If $J \subset I$, $\pm i \notin J$ and $i \in I$, then $A \in \mathcal{A}(J)$ acts by $\xi \mapsto \alpha_{g\overline{h}}(A)\xi$ and its image $\alpha_g(A) \in \mathcal{A}(I)$ acts by $\xi \mapsto \alpha_{\overline{h}}(\alpha_g(A))\xi$.

A similar story holds for the inclusion $J \subset I'$ with $-i \in I'$, but we have to be a bit more careful about the inclusion $J \subset I'$. We decompose it into reflections and inclusions in the *upper* semicircle, $J \to \overline{J} \to \overline{I'} \to I'$. Remembering that (hopefully), $D_h(z \mapsto \overline{z})(A) = \Lambda(F) A^\dagger \Lambda(F)$, the inclusion $J \to I'$ yields

$$A \mapsto \Lambda(F) \left(U_h \Lambda(F) A^\dagger \Lambda(F) U_h^{-1} \right)^\dagger \Lambda(F)$$

which equals $(\Lambda(F)U_h\Lambda(F))A(\Lambda(F)U_h\Lambda(F))^\dagger$. Now $\Lambda(F)U_h\Lambda(F) = U_{\overline{h}}$. This can be seen on the level of generators: $\Lambda(F)\psi_n\Lambda(F) = \psi_{-n}$, so as $E^{ij}(k)$ is a bilinear expression in the fields, we have $\Lambda(F)E(n)\Lambda(F) = E(-n)$, and $U_h = U_{\overline{h}}$. (We may choose h to have winding number zero.)

The inclusion $J \subset I'$ is thus a twist by $\overline{h}$, and the action $\alpha_{g\overline{h}}(A)$ of $A \in \mathcal{A}(J)$ agrees with the action $\alpha_g(\alpha_{\overline{h}}(A))$ of its image $\alpha_{\overline{h}}(A)$ in $\mathcal{A}(I')$.

4.2.1 Vertical multiplication of sectors

Any D-E sector $\mathcal{H}_{DE}$ has a natural structure of $D(S^1_+)$-$E(S^1_+)$-bimodule. The left action is simply given by $D(S^1_+)$, whereas the commuting right action of $E(S^1_+)$ is given by $X : \xi \mapsto \kappa^{-1}E(z \mapsto \overline{z})(X)\kappa\xi$. (Conjugation by κ, well defined on any $\mathbb{Z}/2\mathbb{Z}$-graded Hilbert space, is needed because $E(z \mapsto \overline{z})(X) \in E(S^1_-)$ supercommutes with the left action by X.) For a D-E sector $\mathcal{H}_{DE}$ and an E-F sector $\mathcal{H}_{EF}$, it thus makes sense to consider the $D(S^1_+)$-$F(S^1_+)$ bimodule $\mathcal{H}_{DE} \boxtimes_{E(S^1_+)} \mathcal{H}_{DE}$, which is the bimodule induced (by twisting with κ on F) by a D-F sector. For the D_g-D_h sectors $\mathcal{F}_{g,h}$, we will show that $\mathcal{F}_{f,g} \boxtimes_{\mathcal{A}(S^1_+)} \mathcal{F}_{g,h} \simeq \mathcal{F}_{f,h}$ as a D_f-D_h defect.

We calculate the *vertical* fusion product of $\mathcal{F}_{f,g}$ by $\mathcal{F}_{g,h}$, (As before, we write $\overline{g}$ for $e^{i\phi} \mapsto g(e^{-i\phi})$.) The (left) action of $\mathcal{A}(S^1_+)$ on $\mathcal{F}_{g,h}$ is twisted by h, and because $h|_\cap = 1$ it is not twisted at all. As a left $\mathcal{A}(S^1_+)$-module, we simply have $\mathcal{F} = \mathcal{F}_{g,h}$.

The *right* action of $A \in \mathcal{A}(S^1_+)$ on $\mathcal{F}_{f,g}$ is given by the *left* action of $\kappa^{-1}D_g(z \mapsto \overline{z})(A)\kappa \in \kappa^{-1}\mathcal{A}(S^1_-)\kappa = \mathcal{A}(S^1_+)'$. The left action of $\mathcal{A}(S^1_-)$ on $\mathcal{F}_{f,g}$ is not twisted because the twist f is 1 on S^1_-, and the map $S^1_- \to S^1_+$ maps bicoloured to bicoloured intervals, so it does not introduce a twist either. The right action of $\mathcal{A}(S^1_+)$ on $\mathcal{F}_{f,g}$ is thus simply left multiplication by $\kappa^{-1}\Lambda(F)A^\dagger\Lambda(F)\kappa$.

The Connes Fusion Product is the $\mathcal{A}(S^1)$-bimodule defined as the completion of

$$\mathcal{F}_{f,g} \boxtimes_{\mathcal{A}(S^1_+)} \mathcal{F}_{g,h} := \mathrm{Hom}_{-\,,\mathcal{A}(S^1_+)}(L^2(\mathcal{A}(S^1_+), \mathcal{F}_{f,g}) \otimes \mathcal{F}_{g,h}$$

in the (degenerate!) norm induced by $\langle x \otimes \xi, y \otimes \eta\rangle := \langle (y^\dagger x) \cdot_{\overline{g},h} \xi, \eta\rangle$. Note that $y^\dagger x : L^2(\mathcal{A}(S^1_+)) \to L^2(\mathcal{A}(S^1_+))$ is right-$\mathcal{A}(S^1_+)$ equivariant, and thus multiplication by an element of $\mathcal{A}(S^1_+)$ by the bicommutant theorem. Note also that since both $L^2(\mathcal{A}(S^1_+))$ and $\mathcal{F}_{g,h}$ are $\mathbb{Z}/2\mathbb{Z}$-graded Hilbert spaces, the Hom-space, and hence the CFP, are again graded.

The degeneracy takes care of expected relations like

$$x \otimes a \cdot_{g,h} \xi = x \cdot_{f,g} a \otimes \xi$$

etc. by forcing $\langle x \otimes a \cdot_{g,h} \xi - x \cdot_{f,g} a \otimes \xi, y \otimes \eta\rangle = 0$ for all $y \otimes \eta$.

We use the isomorphism of $\mathcal{A}(S^1_+)$-bimodules $L^2(\mathcal{A}(S^1_+)) \simeq \mathcal{F}$ to see that the requirement that x be an element of $\mathrm{Hom}_{-\,,\mathcal{A}(S^1_+)}(L^2(\mathcal{A}(S^1_+)), \mathcal{F}_{f,g})$ is the same as requiring $x : \mathcal{F} \to \mathcal{F}_{f,g}$ to be an intertwiner of right $\mathcal{A}(S^1_+)$-modules, where the right action on $\mathcal{F}$ is prescribed by Tomita-Takesaki theory, namely

$\xi \mapsto JA^\dagger J\xi$ with $J = \kappa^{-1}\Lambda(F) = \Lambda(F)\kappa$ the Tomita-Takesaki involution for $\mathcal{A}(S^1_+)$, and the right action on $\mathcal{F}_{g,h}$ is prescribed by the net axioms, namely $\xi \mapsto \kappa^{-1}\Lambda(F)A^\dagger\Lambda(F)\kappa$ as explained above. This happens to be the exact same action!

So $x \in \mathrm{Hom}_{-,\mathcal{A}(S^1_+)}(L^2(\mathcal{A}(\cap)), \mathcal{F}_{f,g})$ if and only if $x(Y\xi) = Yx(\xi)$ for $Y = \kappa^{-1}\Lambda(F)A^\dagger\Lambda(F)\kappa = JA^\dagger J$ with $A \in \mathcal{A}(S^1_+)$ arbitrary. Since $A \mapsto JA^\dagger J$ is an antilinear anti-isomorphism by modular theory, it is in particular a surjective map $\mathcal{A}(S^1_+) \to \mathcal{A}(S^1_+)'$. This means that x, considered as an element of $B(\mathcal{F}, \mathcal{F}_{g,h}) \simeq B(\mathcal{F}, \mathcal{F})$, is left multiplication by an element of $\mathcal{A}(S^1_+)'' = \mathcal{A}(S^1_+)$.

Going back to the scalar product, we see that

$$\langle x \otimes \xi, y \otimes \eta\rangle = \langle (y^\dagger x) \cdot_{g,h} \xi, \eta\rangle = \langle x \cdot_{g,h} \xi, y \cdot_{g,h} \eta\rangle \,.$$

In particular, the map $x \otimes \xi \mapsto x \cdot_{g,h} \xi$, is an isometry. The left action of $\mathcal{A}(S^1_+)$ on the fusion product is $a \cdot (x \otimes \xi) := ((a \cdot_{f,g} x) \otimes \xi)$, and the right action is $(x \otimes \xi) \cdot b = x \otimes (\xi \cdot_{g,h} b)$. Because x is right $\mathcal{A}(S^1_+)$ equivariant, these actions are intertwined by the above isometry.

We check that $x \otimes \xi \mapsto$this is also a map of $\mathcal{A}(S^1_+)$-bimodules. The left action of a maps $x \otimes \xi$ to $a \cdot_{f,g} x \otimes \xi$, which goes to $a \cdot_{f,g} \kappa^{-1}x(\xi)$ Because the $\mathcal{A}(S^1_+)$- bimodule structures of the $\mathcal{F}_{g,h}$ are all the same, we may as well label the resulting Hilbert space $\mathcal{F}_{f,h}$.

Proposition 17. *The map* $\phi : \mathcal{F}_{f,g} \boxtimes_{\mathcal{A}(S^1_+)} \mathcal{F}_{g,h} \to \mathcal{F}_{f,h}$ *defined by* $x \otimes \xi \mapsto x(\xi)$ *is an isomorphism* $\mathcal{A}(S^1_+)$*-bimodules.*

All this could be abbreviated by saying that since the right action of $\mathcal{A}(S^1_+)$ on $\mathcal{F}_{f,g}$ and the left action on $\mathcal{F}(g, h)$ are untwisted, the CFP of $L^2(\mathcal{A}(S^1_+))$ by itself is itself again.

We examine the structure of $\mathcal{F}_{f,g} \boxtimes_{\mathcal{A}(S^1_+)} \mathcal{F}_{g,h}$ as a $D_f - D_h$ defect.

- If $A \in \mathcal{A}(I)$ with $\pm i \notin I$, then its action is defined as follows. Suppose that A is a product $A = A_+A_0A_-$ with $A_+ \in \mathcal{A}(S^1_+)$, $A_- \in \kappa\mathcal{A}(S^1_-)\kappa^{-1}$ and $\mathcal{A}_0 \in \mathcal{A}(I_0)$, where I_0 is an interval $\exp(i(-\epsilon, \epsilon)) \cup \exp(i((\pi-\epsilon, \pi+\epsilon)))$ on which f, g, $\overline{g}$ and h are all 1. The action of A_+ (and A_0) is prescribed by the D_f-inclusion of $I \cup S^1_+$ in S^1_+ (which twists by $f\overline{g}$) and the left action of $D_f(S^1_+)$ on the CFP. The action of A_- is prescribed by the D_h inclusion of $I \cup S^1_-$ in S^1_- (which twists by $\overline{h}$) and the right action of $D_h(S^1_+)$ on the CFP, which is made into a left action of $D_h(S^1_-)$ by composition with $D_h(z \mapsto \overline{z})$ and conjugation with κ. (We use the left action of $\kappa D_h(S^1_-)\kappa^{-1}$ directly, so we do not twist by κ.) Then

$$\begin{aligned} x \otimes \xi &\mapsto A_+A_0 \cdot_{f,g} x \otimes A_- \cdot_{g,h} \xi \\ &= U_{f\overline{g}}A_+A_0U^{-1}_{f\overline{g}}x \otimes U_{g\overline{h}}A_-U^{-1}_{g\overline{h}}\xi \\ &= U_fA_+A_0U^{-1}_f x \otimes U_{\overline{h}}A_-U^{-1}_{\overline{h}}\xi \,, \end{aligned}$$

because U_g and $U_{\overline{g}}$ commute with the stuff they surround. Thus

$$\begin{aligned}\phi(A\cdot(x\otimes\xi)) &= U_f A_+ A_0 U_f^{-1} x (U_{\overline{h}} A_- U_{\overline{h}}^{-1}\xi)\\ &= U_f A_+ A_0 U_f^{-1} U_{\overline{h}} A_- U_{\overline{h}}^{-1} x(\xi)\\ &= U_f U_{\overline{h}} A_+ A_0 A_- U_{\overline{h}}^{-1} U_f^{-1} x(\xi)\\ &= U_{f\overline{h}} A U_{f\overline{h}}^{-1}\phi(x\otimes\xi)\end{aligned}$$

because (2nd line) x respects the left $\kappa\mathcal{A}(S^1_-)\kappa^{-1}$-action and because (3rd line) $[U_{\overline{h}}, A_+A_0] = 0$ and $[U_f^{-1}, U_{\overline{h}}A_-] = 0$. In particular, it is clear from this expression that the action of A is indeed an action, and does not depend on the way in which A is decomposed into $A_+A_0A_-$. $\mathcal{A}(I)$ can thus be thought of as acting on $\mathcal{F}$ by $A \mapsto U_{fh}AU_{fh}^{-1}$.

- If $A \in \mathcal{A}(I)$ with $i \in I$, then its action is defined by

$$x\otimes\xi \mapsto A_+A_0 \cdot_{f,g} x \otimes A_- \cdot_{g,h} \xi = U_{\overline{g}} A_+ A_0 U_{\overline{g}}^{-1} x \otimes U_{\overline{h}} A_- U_{\overline{h}}^{-1}\xi\,.$$

Thus

$$\begin{aligned}\phi(A\cdot_{f,g} x\otimes\xi) &= (U_{\overline{g}} A_+ A_0 U_{\overline{g}}^{-1}) x (U_{\overline{h}} A_- U_{\overline{h}}^{-1}\xi)\\ &= A_+ A_0 U_{\overline{h}} A_- U_{\overline{h}}^{-1}\phi(x\otimes\xi)\\ &= U_{\overline{h}} A U_{\overline{h}}^{-1}\phi(x\otimes\xi)\end{aligned}$$

and A can be thought of as having the twisted action $\xi \mapsto U_h A U_h^{-1}\xi$ on $\mathcal{F}$. The twist does nothing in case $I \subset S^1_+$.

- If $A \in \mathcal{A}(I)$ with $-i \in I$, then the action of $A \in \mathcal{A}(I)$ is defined by

$$\begin{aligned}A\cdot(x\otimes\xi) &= A_+A_0 \cdot_{f,g} x\otimes A_- \cdot_{g,h}\xi\\ &= U_f A_+ A_0 U_f^{-1} x \otimes A_-\xi\,,\end{aligned}$$

so that (U_f and x commute with A_-)

$$\begin{aligned}\phi(A\cdot(x\otimes\xi)) &= U_f A_+ A_0 U_f^{-1} x(A_-\xi)\\ &= U_f A_+ A_0 A_- U_f^{-1}\phi(x\otimes\xi)\end{aligned}$$

Note that if $I \subset S^1_-$, then the twist by f does nothing, and $\mathcal{A}(I)$ then has the trivial action on $\mathcal{F}$. In general, the action of $\mathcal{A}(I)$ on $\mathcal{F}$ is given by $A \mapsto U_f A U_f^{-1}$.

Summarising, we see that the D_f-D_h module structure of $\mathcal{F}_{f,g} \boxtimes_{D_g(S^1_+)} \mathcal{F}_{g,h}$ agrees with that of $\mathcal{F}_{f,h}$ for all operators of the form $A_+A_0A_-$ with $A_+ \in \mathcal{A}(S^1_+)$, $A_0 \in \mathcal{A}(I_0)$ and $\mathcal{A}_- \in \mathcal{A}(S^1_+)'$. Since all algebras involved are generated by such elements, the defects must be identical. (We assume that we already know that the thing is a sector.) We have proven

Proposition 18. *The map $\phi : x \otimes \xi \mapsto x(\xi)$ is a unitary isomorphism*

$$\mathcal{F}_{f,g} \boxtimes_{D_g(S^1_+)} \mathcal{F}_{g,h} \to \mathcal{F}_{f,h}$$

of D_f-D_h sectors.

4.2.2 Horizontal multiplication of defects and sectors

In order to state and prove the next propositions, it is convenient to set *and change* a few conventions. If $g \in PO(V)$ with $g|_{S^1_-} = 1$, then g can be uniquely written as $g = g_L g_R = g_R g_L$, where g_L is 1 on the right semicircle $S^1_R := \{z \in S^1 \,;\, \mathrm{Re}(z) > 0\}$ and g_R is 1 on the left semicircle $S^1_L := \{z \in S^1 \,;\, \mathrm{Re}(z) < 0\}$.

We now denote the 'left-right flipped' element by $\overline{g}(x+iy) := g(-x+iy)$. (Before, we used $\overline{g}$ for the 'up-down' flipped element.) The *continuous* loop $g_{RC} := \overline{g}_R g_R$ agrees with g on S^1_R and the *continuous* loop $g_{LC} := g_L \overline{g}_L$ agrees with g on S^1_L. According to proposition 15, $D_g \simeq D_{g'}$ with $g' = g_{RC}^{-1} g = (\overline{g}_R g_R)^{-1} g_L g_R = \overline{g}_R^{-1} g_L$ nontrivial only on the left semicircle S^1_L, and $D_h \simeq D_{h'}$ with $h' = h_{LC}^{-1} h = (h_L \overline{h}_L)^{-1} h_L h_R = \overline{h}_L^{-1} h_R$ nontrivial only on the right semicircle S^1_R. The following proposition says that the composition $D_g * D_h$ of defects is isomorphic to $D_{g'h'}$, and therefore to that given by D_{g*h}, where $g * h := (g'h')_I^{-1} g'h' = \overline{g}_L^{-1} g_R \overline{h}_L^{-1} h_R$. Furthermore, this isomorphism extends to an isomorphism between the sectors $\mathcal{F}_{g,1} * \mathcal{F}_{h,1}$ and $\mathcal{F}_{g*h,1}$.

Proposition 19. *The multiplication of defects is given by $D_g * D_h \simeq D_{g*h}$, where $g * h := \overline{g}_L^{-1} g_R \overline{h}_L^{-1} h_R$ is taken to be 1 on the left semicircle. The identification of defects up to isomorphism with elements of $O(V)$ given by $D_g \mapsto g_i^{-1} g_f$ is therefore an isomorphism of groups. Furthermore, there exists a unitary isomorphism $\phi_{g,h} : \mathcal{F}_{g,1} \boxtimes_{\mathcal{A}(S^1_L)} \mathcal{F}_{h,1} \to \mathcal{F}_{g*h,1}$ that intertwines the $D_g * D_h$ defect $\mathcal{F}_{g,1} * \mathcal{F}_{h,1}$ with the D_{g*h} defect $\mathcal{F}_{g*h,1}$.*

Proof. We construct a representation of $D_g * D_h(J)$ for a bicoloured interval J. Let I be the left semicircle $I = S^1_L := \{e^{i\phi} \,;\, \phi \in [-\pi/2, -3\pi/2]\}$ and let $\overline{I} = S^1_R$ be its complex conjugate, the right semicircle. We define the bicoloured intervals $[J_W] := J_W \cup_i S^1_R$ and $[J_B] = S^1_L \cup_i J_B$, coloured such that $J_W \subset [J_W]$ is the white part of $[J_W]$ and $J_B \subset [J_B]$ the black part of $[J_B]$. (In words, we attach a black right semicircle to the left, white part of J and we attach a left, white semicircle to the right, black part of J.) We deviate from the definition of [BDH09] in that we do not attach a little 'buffer interval' between the semicircles and the intervals. Thus S^1_L maps to $[J_B]$ by inclusion ι and to $[J_W]$ by $s : z \mapsto -\overline{z}$.

We choose the faithful representation $\mathcal{F}_g$ of $D_g([J_W])$, on which an element $a \in D_g([J_W]) \simeq \mathcal{A}([J_W])$ acts by $\xi \mapsto a\xi$. Likewise $\mathcal{F}_h$ for $D_h([J_W])$. This introduces a left action of $\mathcal{A}(S^1_L)$ on $\mathcal{F}_h$ by $a : \xi \mapsto D_h(\iota)(a)\xi = \alpha_h(a)\xi$ and a commuting right action of $\mathcal{A}(S^1_L)$ on $\mathcal{F}_g$ by $a : \xi \mapsto D_g(z \mapsto -\overline{z})(\kappa a \kappa^{-1})\xi = \alpha_g(\kappa^{-1}\Lambda(S)a^\dagger\Lambda(S)\kappa)\xi$, κ commutes with the even U_g's, which happens to equal $\alpha_g(Ja^\dagger J)$. We may thus think of $\mathcal{F}_g$ as the D_g-D_1 sector $\mathcal{F}_{g,1}$ and of $\mathcal{F}_h$ as the D_h-D_1 sector $\mathcal{F}_{h,1}$.

We calculate $\mathcal{F}_g \boxtimes_{\mathcal{A}(S^1_L)} \mathcal{F}_h$. This is by definition the closure of the tensor product

$$\mathrm{Hom}_{-,\mathcal{A}(S^1_L)}(L^2(\mathcal{A}(S^1_L)), \mathcal{F}_g) \otimes \mathcal{F}_h$$

w.r.t. the degenerate inner product

$$\langle x \otimes \xi, y \otimes \eta \rangle := \langle (y^\dagger x) \cdot_h \xi, \eta \rangle$$

where we used the fact that $y^\dagger x : L^2(\mathcal{A}(S^1_L)) \to L^2(\mathcal{A}(S^1_L))$ is right-$\mathcal{A}(S^1_L)'$ equivariant and must therefore be leftmultiplication by an element $a \in \mathcal{A}(S^1_L)$. This element acts from the left on $\mathcal{F}_h$ by the twisted action $\xi \mapsto \alpha_h(a)\xi$, and this is what we mean by $(y^\dagger x) \cdot_h \xi$.

We identify $L^2(\mathcal{A}(S^1_L)) = \mathcal{F}$ as an $\mathcal{A}(S^1_L)$ bimodule in the usual way, $[a] \mapsto a\Omega$, so that the left action is $a \cdot \xi = a\xi$ and the right action $\xi a = Ja^\dagger J\xi$. The degeneracy of the tensor product takes care of the relations $xa \otimes \xi = x \otimes a \cdot_h \xi$, where the right action of a on x is given by $xa : \xi \mapsto x(a\xi)$. Indeed, we have

$$\langle xa \otimes \xi, y \otimes \eta \rangle = \langle (y^\dagger xa) \cdot_h \xi, \eta \rangle = \langle (y^\dagger x) \cdot_h (a \cdot_h \xi), \eta \rangle = \langle x \otimes a \cdot_h \xi, y \otimes \eta \rangle$$

for all $y \otimes \eta \in \mathcal{F}_g \boxtimes_{\mathcal{A}(S^1_L)} \mathcal{F}_h$.

We prove that x is an element of $\mathrm{Hom}_{-,\mathcal{A}(S^1_L)}(L^2(\mathcal{A}(S^1_L)), \mathcal{F}_g)$ if and only if $xU_g^{-1} =: x_0 \in \mathcal{A}(S^1_L)$. (We identify $\mathcal{F}_g \simeq \mathcal{F}$ as a Hilbert space but not as an $\mathcal{A}(S^1_L)$ module. By $x_0 \in \mathcal{A}(S^1_L)$ we mean the map $\mathcal{F} \to \mathcal{F}_g : \xi \mapsto x_0\xi$, not the map $\xi \mapsto \alpha_g(a)\xi$.)

By right $\mathcal{A}(S^1_L)$-invariance, $x(Ja^\dagger J\xi) = \alpha_g(Ja^\dagger J)x(\xi)$ for all $a \in \mathcal{A}(S^1_L)$, and therefore $x(b\xi) = \alpha_g(b)x(\xi)$ for all $b \in J\mathcal{A}(S^1_L)J = \kappa^{-1}\mathcal{A}(S^1_R)\kappa = \mathcal{A}(S^1_L)'$. Thus x is a morphism of left-$\mathcal{A}(S^1_L)'$ modules. Now let $U_{g,R}$ be a lift of the *continuous* loop g_{RC} that coincides with g on S^1_R. Because $U_{g,R}\mathcal{A}(S^1_R)U_{g,R}^{-1} = \mathcal{A}(S^1_R)$ and $\kappa U_{g,R}\kappa^{-1} = U_{g,R}$ (the Lie algebra generators are even), we have $U_{g,R}\mathcal{A}(S^1_L)'U_{g,R}^{-1} = \mathcal{A}(S^1_L)'$ for all $b \in \mathcal{A}(S^1_L)'$:

$$\begin{aligned} xU_{g,R}^{-1}(b\xi) &= x(U_{g,R}^{-1}bU_{g,R}U_{g,R}^{-1}\xi) \\ &= U_{g,R}^{-1}bU_{g,R} \cdot_g x(U_{g,R}^{-1}\xi) \\ &= bx(U_{g,R}^{-1}\xi)\,. \end{aligned}$$

This means that $x_0 = xU_{g,R}^{-1} \in B(\mathcal{F})$ commutes with all $b \in \mathcal{A}(S^1_L)'$, so $x_0 \in \mathcal{A}(S^1_L)'' = \mathcal{A}(S^1_L)$. Now $g_{RC} = \overline{g}_R g_R$, as explained above. A different choice of g_{RC}, agreeing with g on S^1_R, would have yielded a different x_0, but still one in $\mathcal{A}(S^1_L)$.

As announced, we look for an element a such that $y^\dagger x(\xi) = a \cdot \xi$ in $L^2(\mathcal{A}(S^1_L))$. Since $y^\dagger x = U_{g,R}^{-1} y_0^\dagger x_0 U_{g,R} \in \mathcal{A}(S^1_L)$, this is just $a = y^\dagger x$. We thus have

$$\langle x \otimes \xi, y \otimes \eta \rangle = \langle U_{h,L} y^\dagger x U_{h,L}^{-1}\xi, \eta \rangle = \langle xU_{h,L}^{-1}\xi, yU_{h,L}^{-1}\eta \rangle = \langle U_{g,L}^{-1}xU_{h,L}^{-1}\xi, U_{g,L}^{-1}yU_{h,L}^{-1}\eta \rangle$$

where $U_h := U_{h,L}$ is a the lift of h_{LC} which agrees with h on S^1_L. This shows that

$$\mathcal{F}_g \boxtimes_{\mathcal{A}(S^1_L)} \mathcal{F}_h \to \mathcal{F} \quad : \quad x \otimes \xi \mapsto U_{g,L}^{-1} x U_{h,L}^{-1} \xi$$

is an isometry of (pre)-Hilbert spaces $\mathcal{F}_g \boxtimes_{\mathcal{A}(S^1_L)} \mathcal{F}_h \to \mathcal{F}$.

The algebra $D_g * D_h(J)$ is, by definition, the v.N. algebra generated by $\mathcal{A}(J_B)$ and $\mathcal{A}(J_W)$, acting by $a : x \otimes \xi \mapsto x \otimes a\xi$ and $a : x \otimes \xi \mapsto ax \otimes \xi$ respectively. The latter action amounts to

$$\begin{aligned} a : U_{g,L}^{-1} x U_{h,L}^{-1} \xi &\mapsto U_{g,L}^{-1} a x U_{h,L}^{-1} \xi \\ &= (U_{g,L}^{-1} a U_{g,L}) U_{g,L}^{-1} x U_{h,L}^{-1} \xi \end{aligned}$$

so that the image of $a \in \mathcal{A}(J_W)$ in $D_g * D_h(J)$ naturally acts through a twist by g_{LC}^{-1}.

The former action is given by

$$\begin{aligned} a : U_{g,L}^{-1} x U_h^{L\,-1} \xi &\mapsto U_{g,L}^{-1} x U_{h,L}^{-1} a \xi \\ &= U_{g,L}^{-1} x_0 (U_{g,R} U_{h,L}^{-1}) a (U_{g,R} U_{h,L}^{-1})^{-1} U_{g,R} U_{h,L}^{-1} \xi \\ &= U_{g,L}^{-1} (U_{g,R} U_{h,I}^{-1}) a (U_{g,R} U_{h,L}^{-1})^{-1} x_0 U_{g,R} U_{h,L}^{-1} \xi \\ &= (U_{g,L}^{-1} U_{g,R} U_{h,L}^{-1}) a (U_{g,L}^{-1} U_{g,R} U_{h,L}^{-1})^{-1} U_{g,L}^{-1} x U_{h,L}^{-1} \xi \end{aligned}$$

(note that $x_0 \in \mathcal{A}(S^1_L)$ commutes with $(U_{g,R} U_{h,L}^{-1}) a (U_{g,R} U_{h,L}^{-1})^{-1} \in \mathcal{A}(J_B)$). The natural way for a to act is thus through a twist by $g_{LC}^{-1} g_{RC} h_{LC}^{-1}$.

The algebra $D_g * D_h(J)$ is thus simply

$$U_{g,I}^{-1} \mathcal{A}(J_W) U_{g,I} \vee U_{h,I}^{-1} \mathcal{A}(J_B) U_{h,I} = \mathcal{A}(J_W) \vee \mathcal{A}(J_B) = \mathcal{A}(J)\,,$$

and its action on $\mathcal{F}$ is the ordinary one, $a : \xi \mapsto a\xi$.

If $K \subset J_W$, then the inclusion $D_g(K) \hookrightarrow D_g * D_h(J)$ is determined by its action on $\mathcal{F}_g \boxtimes_{\mathcal{A}(S^1_L)} \mathcal{F}_h$, which is given by $D_g(K) \ni a : x \otimes \xi \mapsto \alpha_g(a) x \otimes \xi$. By the previous calculation, this yields the action $a : \xi \mapsto U_{g,I}^{-1} U_{g,K} a U_{g,K}^{-1} U_{g,I} \xi$. The white inclusion is thus twisted not by g, but by $g_{LC}^{-1} g$, where g_{LC} is a continuous loop in $O(V)$ that agrees with g on I. With $g_{LC} = g_L \overline{g}_L$, we see that the twist is by $\overline{g}_L^{-1} g_R$. Since this is the identity on $K \subset J_W$, the white action is not twisted at all!

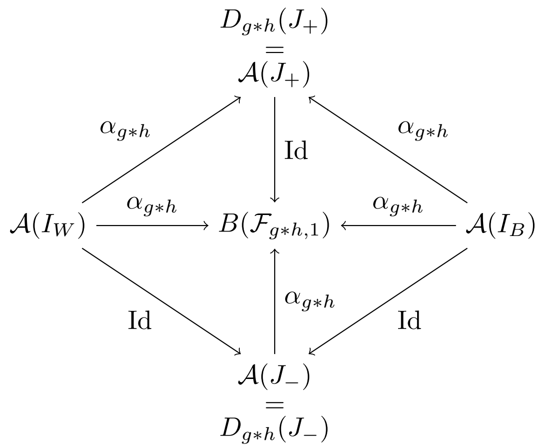
If $K \subset J_B$, then the inclusion $D_g(K) \hookrightarrow D_g * D_h(J)$ is determined by the action on $\mathcal{F}_g \boxtimes_{\mathcal{A}(S^1_L)} \mathcal{F}_h$ given by $D_g(K) \ni a : x \otimes \xi \mapsto x \otimes \alpha_h(a)\xi$. On $\mathcal{F}$, this looks like $a : \xi \mapsto U a U^{-1} \xi$ with $U := U_{g,L}^{-1} U_{g,R} U_{h,L}^{-1} U_{h,K}$. Because $K \subset S^1_R$, we have $g_{RC} = g$ on K, so the black inclusion is twisted by $g_{LC}^{-1} g h_{LC}^{-1} h = \overline{g}_L^{-1} g_R \overline{h}_L^{-1} h_R$.

Since this function is 1 on I, we conclude that *both the white and the black inclusion* are twisted by $g * h := \overline{g}_L^{-1} g_R \overline{h}_L^{-1} h_R$.

The equivalence class of $g \in PO(V)$ under the (left) action by $\Omega O(V)$ is given by $g_i^{-1} g_f \in O(V)$, with $g_i := \lim_{z \to i, \mathrm{Re}(z) < 0} g(z)$ and $g_f := \lim_{z \to i, \mathrm{Re}(z) > 0} g(z)$. The equivalence class of h is given by $h_i^{-1} h_f$. The equivalence class of $g_{RC}^{-1} g h_{LC}^{-1} h$ is $(g_f^{-1} g_i)^{-1} (h_i^{-1} h_f) = g_i^{-1} g_f h_i^{-1} h_f$, so that the map $PO(V) \to O(V) : g \mapsto g_i^{-1} g_f$ factors through a group isomorphism $\{[D_g]\,;\, g \in PO(V)\} \to O(V)\,.$

It is now easy to see that the unitary map $\mathcal{F}_{g,1} \boxtimes \mathcal{F}_{h,1} \to \mathcal{F}_{g*h,1}$ is a map of sectors that covers the natural transformation $D_g * D_h \to D_{g*h}$ given above, as well as the identity morphism $D_1 \to D_1$. (The natural transformation is defined by setting $N(J) : D_g * D_h(J) \to D_{g*h}(J)$ to the identity when both sides are identified with $\mathcal{A}(J)$ for 2-coloured J, and by setting $N(I) : D_g * D_h(I) \simeq \mathcal{A}(I) \to \mathcal{A}(I) \simeq D_{g*h}(I)$ to be $a \mapsto \alpha_{g*h}(a)$.)

For 2-coloured $J_+ \ni i$ and $J_- \ni -i$ and 1-coloured $I_W \subset J_+ \cap J_-$ and $J_B \subset J_+ \cap J_-$, commutativity of the following diagram



and its compatibility with the constructed map $\phi_{g,h} : \mathcal{F}_{g,1} \boxtimes \mathcal{F}_{h,1} \to \mathcal{F}_{g*h,1}$ and the natural transformations $D_g * D_h \to D_{g*h}$ and identity $D_1 \to D_1$, show that $\phi_{g,h}$ is an isomorphism between the $D_g * D_h$-D_1 sector $\mathcal{F}_{g,1} * \mathcal{F}_{h,1}$ and the $D_{g*h} - D_1$ defect $\mathcal{F}_{g*h}$. □

Remark 9. Note that the map $PO(V) \to O(V) : g \mapsto g_i^{-1} g_f$ that yields the group structure on the group $\{[D_g]\,;\, g \in PO(V)\}$ is not itself a group homomorphism! It does however constitute a homomorphism when restricted to the based paths $P_e O(V)$.

5 2-Groups and 3-Categories

We slightly alter the definition of the 3-category CN3 of conformal nets in [DH12].

5.1 The 3-category of nets

The objects of CN3 will be conformal nets as before. A 1-morphism between 2 conformal nets $\mathcal{A}$ and $\mathcal{B}$ will be a *class* ${}_{\mathcal{A}}[D]_{\mathcal{B}}$ of $\mathcal{A} - \mathcal{B}$-defects, where 2 defects D and E are deemed equivalent if and only if there exists an invertible natural transformation $N : D \to E$ (both D and E are Int-vNAlg functors). Recall that part of the definition of a defect D was a pair of natural equivalences

$N^D_{black} : \mathcal{A} \to D|_{black}$ and $N^D_{white} : \mathcal{B} \to D|_{white}$ between the nets $\mathcal{A}$ and $\mathcal{B}$ and the restrictions of D to the black and white intervals respectively. The natural transformation $N : D \to E$ is required to respect this: $NE_{black}(I) = N(I) \circ N^D_{black}(I)$ for black intervals and $N^E_{white}(I) = N(I) \circ N^D_{white}(I)$ for white ones. Note that because of this restriction, the action of $N(I) : D(I) \to E(I)$ is already determined on black and white intervals, and because the images of these generate $D(J)$ and $D'(J)$ for the bicoloured intervals, there exists at most one natural transformation between D and D'.

A 2-morphism $\mathcal{F}$ between two $\mathcal{A}$-$\mathcal{B}$ 1-morphisms $[D]$ and $[E]$ will be a $\mathbb{Z}/2\mathbb{Z}$-graded Hilbert space $\mathcal{F}$, equipped with the structure ρ^{DE} of D-E sector for each two elements $D \in [D], E \in [E]$. If $D' \in [D], E' \in [E]$ is another choice of representing elements, then the (unique) natural transformations $N_D : D \to D'$ and $N_E : E \to E'$ are required to intertwine the structures of D-E sector ρ^{DE} and D'-E' sector $\rho^{D'E'}$ living on $\mathcal{F}$, i.e. $\rho^{DE}_{J_+} = \rho^{D'E'}_{J_+} \circ N_D(J_+)$ for bicoloured J_+ with $-i \notin J_+$ and $\rho^{DE}_{J_-} = \rho^{D'E'}_{J_-} \circ N_E(J_-)$ for bicoloured J_- with $i \notin J_-$, and $\rho^{DE}_I = \rho^{D'E'}_I$ for the representations of $\mathcal{A}(I)$ and $\mathcal{B}(I)$ for the black and white intervals respectively. A 2-morphism $\mathcal{F}$ is fully specified by its structure of D-E sector for a single choice of $D \in [D]$, $E \in [E]$.

A 3-morphism U between two $[D]$-$[E]$ sectors $\mathcal{F}$ and $\mathcal{G}$ is an invertible isometry $U : \mathcal{F} \to \mathcal{G}$ of $\mathbb{Z}/2\mathbb{Z}$-graded Hilbert spaces that intertwines the D-E sector ρ^{DE} on $\mathcal{F}$ with the D-E sector τ^{DE} on G for one choice (and therefore for all choices) of $D \in [D]$, $E \in [E]$.

5.2 The weak 2-group $G(V)$

Following [DH12], we define the weak 2-group $G(V)$ as follows. The objects are invertible $[D_G]$-$[D_1]$ sectors, i.e. invertible 2-morphisms $([D_G], \beta)$, in the 3-category of conformal nets, between $[D_G]$ and $[D_1]$.

$$G_0(V) := \bigcup_{G \in SO(V)} \mathrm{Iso}^2([D_G], [D_1])$$

The class of defects $[D_G]$ is the one determined by piecewise C^1 paths that are $\neq 1$ only on the right upper quadrant of the circle, satisfy $g(1) = 1 \in SO(V)$ and $g(i) = G \in SO(V)$, and are continuous on $S^1 - \{i\}$. The class $[D_1]$ contains the free fermionic field. The multiplication $G_0(V) \times G_0(V) \to G_0(V)$ (which is associative only up to isomorphism) is given by horizontal fusion of defects and sectors. We have seen that two defects are equivalent, $[D_g] = [D_{g'}]$, if and only if $gg'^{-1} \in \Omega(SO(V))$, and also that $G_0(V) \to SO(V)$ respects the (weak) group structure.

The morphisms in $G(V)$ are the invertible 3-morphisms between the sectors.

$$\mathrm{Hom}(([D_g], \beta), ([D_{g'}], \beta')) = \mathrm{Iso}^3(\beta, \beta')\,.$$

Multiplication is by horizontal fusion, and the groupoid structure is clear. We will see that every D_g-D_1 sector is isomorphic to one of the form $\mathcal{F}_{g,1} \hat{\otimes} \mathbb{C}[0]$ or the same with a degree shift, $\mathcal{F}_{g,1}\hat{\otimes}\mathbb{C}[1]$. Also, we will see that $\mathcal{F}_{g,1}$ is isomorphic to $\mathcal{F}_{g',1}$ if and only if $gg'^{-1} \in \Omega\mathrm{Spin}(V)$. In that case, the 3-morphisms are all multiples of a single unitary operator. Thus

$$\mathrm{Hom}(([D_g], \mathcal{F}_{g,1}), ([D_{g'}], \mathcal{F}_{g',1})) \simeq \begin{cases} U(1) & \text{if} \quad gg'^{-1} \in \Omega\mathrm{Spin}(V) \\ \emptyset & \text{if} \quad gg'^{-1} \notin \Omega\mathrm{Spin}(V) \end{cases}$$

In particular, the orbit space $G_0(V)/G_1(V)$ is Spin(V).

Remark 10. It seems natural to study, instead of $G(V)$, the 2-category of automorphisms of $\mathrm{Fer}(V)$. Its objects are the classes of defects $[D_g]$, its morphisms the invertible D_f-D_g sectors (e.g. $[\mathcal{F}_{f,g}]$) and its 2-morphisms the unitary intertwiners between them. Instead of paths that live on the right upper quadrant and are allowed to jump at i, one would then use paths that live on the right semicircle and are allowed to jump at $\{i, -i\}$.

Remark 11. Using the left-right reflection and the modular operators w.r.t. $\mathcal{A}(S1_+)$, it is hopefully possible to give $G(V)$ the structure of a *coherent* 2-group. If we are willing to restrict the sectors a bit, say by fixing a manifold, vector bundle and connection (M, E, ∇) and considering all the sectors that arise by pulling back (E, ∇) along a parameterised loop $S^1 \to M$, then one may also cherish hope of describing $G(V)$ as some kind of topological weak 2-group.

5.3 Equivalence classes of sectors

The following corollary to proposition 18 is immediate:

Corollary 20. *The $D_g - D_h$ sector $\mathcal{F}_{g,h}$ is invertible.*

Proof. We have $\mathcal{F}_{g,h} \boxtimes \mathcal{F}_{h,g} \simeq \mathcal{F}_{g,g}$ and $\mathcal{F}_{h,g} \boxtimes \mathcal{F}_{g,h} \simeq \mathcal{F}_{h,h}$. We need only show that these are the respective identity defects for D_g and D_h. The identity defect of D_g is $L^2(D_g(S^1_+)) \simeq \mathcal{F}$. According to the proof of proposition 14, the action of $a \in D_g(I)$ with $i \in I$, $I \subseteq S^1_+$ is by $\pi(a)$, the action of $b \in D_g(I)$ with $-i \in I$, $I \subseteq S^1_-$ is by $\pi(b)$, and the action of $a \in D_g(I)$ with $\pm i \notin I$ is by $\pi(\alpha_{g\overline{g}})(a)$. (Here, $\overline{g}$ means the up-down flip.) This coincides with the definition of $\mathcal{F}_{g,\overline{g}}$. □

Proposition 21. *Every invertible D_g-D_h sector $\mathcal{F}$ is equivalent to either $\mathcal{F}_{g,h}$ or to $\mathcal{F}_{g,h}$ with a degree shift. In other words, $\mathcal{F} \simeq \mathcal{F}_{g,h}\otimes\mathbb{C}[0]$ or $\mathcal{F} \simeq \mathcal{F}_{g,h}\otimes\mathbb{C}[1]$.*

Proof. Since D_g and D_h are invertible defects of an invertible conformal net $\mathcal{A} = \mathrm{Fer}(V)$, its 1-category of sectors (objects) and unitaries (1-morphisms) is isomorphic to the 1-category $\mathrm{Aut}(1_1, 1_1)$ of sectors and unitaries of the identity defect of the identity conformal net. By 3.19 of [DH12], this is equivalent to the 1-category of $\mathbb{Z}/2\mathbb{Z}$ graded $\mathbb{C} - \mathbb{C}$ bimodules. The invertible objects here are $\mathbb{C}[0]$ and $\mathbb{C}[1]$, q.e.d.

There should be a more insightful proof that bypasses the invertibility of $\mathrm{Fer}(V)$ (but not of the defects). Again, since $\mathcal{F}_{1,g}$ and $\mathcal{F}_{h,1}$ are invertible D_1-D_g

and D_h-D_1 sectors, we may form the vertical product $\mathcal{F}_{1,g} \boxtimes_{\mathcal{A}(S^1_+)} \mathcal{F} \boxtimes_{\mathcal{A}(S^1_+)} \mathcal{F}_{h,1}$ and assume that $\mathcal{F}$ is an invertible D_1-D_1 defect without loss of generality.

A D_1-D_1 defect $\mathcal{F}$ is a set of compatible $\mathcal{A}(I)$-representations ρ_I for all intervals $I \subset S^1$ that do not contain both i and $-i$. The images $\rho_I(\mathcal{A}(I))$ and $\rho_J(\mathcal{A}(J))$ supercommute when I and J have at most a point in common.

In the particular case of $\mathcal{A} = \mathrm{Fer}(V)$, we even have $\mathcal{A}(S^1_+) = \mathcal{A}(S^1_-)^{\mathrm{gr}\prime}$ inside $\mathcal{A}(S^1)$, so that $\mathcal{A}(S^1_+) \vee \mathcal{A}(S^1_-) = (\mathcal{A}(S^1_+)^{\mathrm{gr}\prime} \cap \mathcal{A}(S^1_-)^{\mathrm{gr}\prime})^{\mathrm{gr}\prime} = \mathbb{C}^{\mathrm{gr}\prime} = \mathcal{A}(S^1)$. The homomorphisms $\rho_{S^1_+}$ and $\rho_{S^1_-}$ thus define a representation of the *algebraic* (super) tensor product $\mathcal{A}(S^1_+)\hat{\otimes}\mathcal{A}(S^1_-)$, which is a dense subalgebra of $\mathcal{A}(S^1)$. One should then prove that this representation extends (uniquely of course) to the closure $\mathcal{A}(S^1)$, presumably by using the fact that we already know that it extends to every $\mathcal{A}(I) \subset \mathcal{A}(S^1)$.

(This I didn't manage.)

Now $D_1 - D_1$ defects $\mathcal{F}$ correspond to $\mathcal{A}(S^1)$-representations, and those are easily classified. If we forget the grading, then we have $\mathcal{A}(S^1) \simeq B(\mathcal{F}_{1,1})$, so that every $\mathcal{A}(S^1)$-representation is unitarily equivalent to $\mathcal{F}_{1,1} \otimes \mathcal{H}$, where the action is by $a \mapsto a \otimes 1$. Since $\mathcal{F}$ must be an invertible sector, $\mathcal{H}$ must be 1-dimensional. Thus $\mathcal{F}$ is isomorphic to $\mathcal{F}_{1,1}$ as a (non-graded) v.N. algebra representation.

Now we remember that $\mathcal{A}(S^1) = B(\mathcal{F}_0 \oplus \mathcal{F}_1)$ as a $\mathbb{Z}/2\mathbb{Z}$ graded von Neumann algebra, and that $\mathcal{F}$ is also graded, $\mathcal{F} = V_0 \oplus V_1$. Let Q_0 be the orthogonal projection onto V_0 and $Q_1 = \mathbf{1} - Q_0$ the one onto V_1. Since the image of $\mathcal{A}(S^1)$ is all of $B(\mathcal{F})$, we can consider Q_0 and Q_1 as elements of $\mathcal{A}(S^1)$, and because they respect the grading in the representation, they must be even. Also, since the even part $\mathcal{A}(S^1)_0$ respect the grading, it commutes with Q_0 and Q_1. Thus $Q_{0,1} \in \mathcal{A}_0^{S^1} \cap \mathcal{A}(S^1)'_0$. Identifying $\mathcal{A}(S^1) \simeq B(\mathcal{F}_0)\hat{\otimes}M(1|1)$, we see that $Q_{0,1} = 1 \otimes q_{0,1}$ with $q_{0,1}$ a complementary pair of even projections in $M(1|1)$. Thus there are 2 options: either Q_0 is the projection onto $\mathcal{F}_0$ and Q_1 onto $\mathcal{F}_1$, or Q_0 is the projection onto $\mathcal{F}_1$ and Q_1 onto $\mathcal{F}_0$. □

5.4 Equivalence classes of defects and sectors in $\mathrm{Fer}(V)$

Let $P_eO(V)$ be the group of continuous maps $g : [0,1] \to O(V)$ such that $g(0) = 1$. We consider $P_eO(V)$ as a subgroup of $PO(V)$ by identifying $g \in P_eO(V)$ with the element $\tilde{g} \in PO(V)$ defined by $\tilde{g}(e^{i\phi}) = g(2\phi/\pi)$ for $\phi \in [0, \pi/2]$ and $\tilde{g}(e^{i\phi}) = 1$ for $\phi \in (\pi/2, 2\pi]$.

Given a graded Hilbert space $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$, we denote by $PU(\mathcal{H}_0 \oplus \mathcal{H}_1) \simeq (U(\mathcal{H}_0) \times U(\mathcal{H}_1))/S^1$ the group of projective unitaries preserving the grading. Given a pair $(p, V) \in P_eO(V) \times PU(\mathcal{F}_0 \oplus \mathcal{F}_1)$, we define an element $([D_g], \mathcal{F}^V_{g,1})$ of $G(V)$. The graded Hilbert space $\mathcal{F}^V_{g,1}$ is just $\mathcal{F}$. The homomorphisms $\rho^{g,V}_I : D_g(I) \to B(\mathcal{F}^V_{g,1})$ defining the D_g-D_1 sector $\mathcal{F}^V_{g,1}$ are given by $\mathcal{F}_{g,1}$ for $V = \mathrm{Id}$ and by

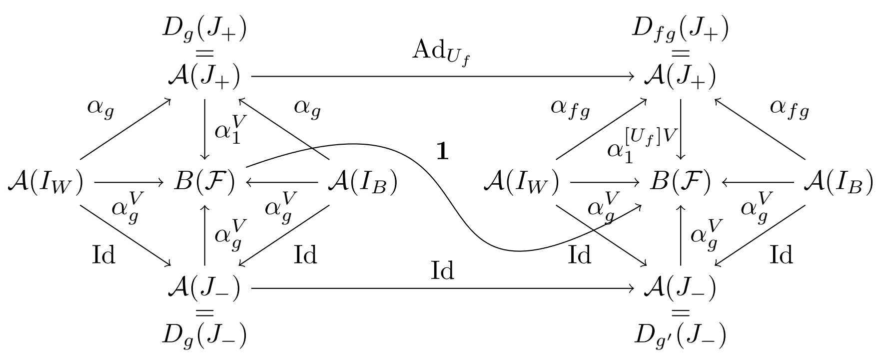
$$\rho^{g,V}_I := \mathrm{Ad}_{V^{-1}} \circ \rho^{g,\mathrm{Id}}_I$$

for general V. Recall that $[D_g] = [D_{g'}]$ if and only if $g' = fg$ with $f \in \Omega O(V) \cap P_e O(V) = \Omega_e O(V)$, in which case $[D_g] = [D_{g'}] = [D_G]$ for $G = g(1) = g'(1)$. It therefore makes sense to ask whether or not $([D_g], \mathcal{F}^V_{g,1})$ and $([D_{g'}], \mathcal{F}^{V'}_{g',1})$ define the same 2-morphism, and whether or not they define isomorphic 2-morphisms.

Proposition 22. *The $[D_g]$-$[D_1]$ sector $\mathcal{F}^V_{g,1}$ and the $[D_{g'}]$-$[D_1]$ sector $\mathcal{F}^{V'}_{g',1}$ are isomorphic if and only if $g' = fg$ for some $f \in \Omega_e \mathrm{Spin}(V)$. The grading-preserving isomorphisms (unique up to S^1) are given by the elements in $U(\mathcal{F}_0 \oplus \mathcal{F}_1)$ representing $V'^{-1}[U_f]V \in PU(\mathcal{F}_0 \oplus \mathcal{F}_1)$. The two sectors are equal if and only if the isomorphism is the identity, i.e. when $V' = [U_f]V$.*

Proof. The requirement that $g = fg'$ with $f \in \Omega_e O(V)$ follows from $[D_g] = [D_{g'}]$ as noted in proposition 15. The unique invertible natural transformations $D_1 \to D_1$ and $D_g \to D'_g$ are the identity $N : D_1 \to D_1$ and the natural transformation $N : D_g \to D_{g'}$ given by $N(I) = \mathrm{Id}$ for $i \notin I$ and $N(J) = \mathrm{Ad}_{U_f}$ for $i \in J$. The sector $\mathcal{F}^V_{g,1}$ defines actions of $D_{g'}(J_+)$ and $D_1(J_-)$ on $\mathcal{F}$ through the natural transformation.

In the following diagram, the l.h.s. represents the actions of D_g and D_1 on the $\mathcal{F}^V_{g,1}$ sector, while the r.h.s. represents the corresponding actions of $D_{g'}$ and D_1. For brevity, we write α^V_g for $\mathrm{Ad}_{V^{-1}} \circ \alpha$.



Since

$$\alpha^V_g(X) = V^{-1} U_g X U_g^{-1} V = (U_f V)^{-1} U_{fg} X U_{fg}^{-1} (U_f V) = \alpha^{[U_f]V}_{fg}$$

we see that the induced action of $D_{g'}$ on $\mathcal{F}$ is the one belonging to $\mathcal{F}^{U_f V}_{fg}$. The sectors $\mathcal{F}^V_g$ and $\mathcal{F}^{[U_f]V}_{fg}$ therefore define the *same* $[D_G]$-$[D_1]$ 2-morphism.

If, in the above diagram, we replace the identity $\mathbf{1} : B(\mathcal{F}) \to B(\mathcal{F})$ by a unitary operator $\mathrm{Ad}_U : B(\mathcal{F}_{g,1}) \to B(\mathcal{F}_{g',1})$, then we see that the corresponding unitary isomorphism $U : \mathcal{F} = \mathcal{F}_{g,1} \to \mathcal{F}_{g',1} = \mathcal{F}$ is an invertible 3-morphism between the classes of sectors defined by the D_g-D_1 sector $\mathcal{F}^V_{g,1}$ and the D_{fg}-D_1 sector $\mathcal{F}^{[U_f]V[U^{-1}]}_{fg,1}$. With $[U] = [V'^{-1} U_f V]$, this yields the required result. Because U, V and V' are even operators, so is U_f. Because of proposition 13, we must therefore have $f \in \Omega_e \mathrm{Spin}(V)$. $\square$

This means that the defects $[D_g]$ are classified by $g_f \in SO(V)$, the isomorphism classes of 2-morphisms by $(g_f, [\pi_1(g)]) \in \mathrm{Spin}(V)$, and, because $P_e O(V) = P_e \mathrm{Spin}(V)$, the 2-morphisms by

$$[g, V] \in \left(P_e\mathrm{Spin}(V) \times U(\mathcal{F}_0 \oplus \mathcal{F}_1)\right)/\Omega_e \mathrm{Spin}(V) \times S^1 .$$

5.5 The model of Stolz-Teichner

Strictly speaking, the 2-group $G(V)$ is not strict. In order to fix this, we attempt to identify the horizontal fusion product $[\mathcal{F}_f^U] * [\mathcal{F}_g^V] = [\mathcal{F}_f^U \boxtimes_{\mathcal{A}(S^1_L)} \mathcal{F}_g^V]$ with a sector that is again of the form $[\mathcal{F}_h^W]$. This turns out to be easier (and sufficient for our immediate goal, which is to provide a weak equivalence of 2-groups) if we restrict U and V to unitary operators in $\mathcal{A}(S^1_R)$ Recall that we already required U and V to be even operators.

Proposition 23. *Let $\overline{U}, \overline{V} \in PU(\mathcal{A}_{\mathrm{ev}}(S^1_R))$ be even (i.e. grading preserving) projective unitary operators, and $U \in \overline{U}, V \in \overline{V}$. Let $f, g \in P_e O(V)$. Then the map $\mathcal{F}_f^U \boxtimes_{\mathcal{A}(S^1_L)} \mathcal{F}_g^V \to \mathcal{F}_{fg}^{\alpha_g(V)U}$ defined by*

$$x \otimes \xi \mapsto x(\xi)$$

is an isomorphism of sectors that extends to an isomorphism $[\mathcal{F}_f^U] * [\mathcal{F}_g^V] \to [\mathcal{F}_{fg}^{\alpha_g(V)U}]$ on 2-morphisms. Under this isomorphism, the fusion product $X * Y$ of $X : \mathcal{F}_f^U \to \mathcal{F}_f^{UX^{-1}}$ and $Y : \mathcal{F}_g^V \to \mathcal{F}^{VY^{-1}}$ is

$$X * Y = XU^{-1}\alpha_f(Y)U .$$

Proof. Note that for $x \in \mathrm{Hom}_{-,\mathcal{A}(S^1_R)}(\mathcal{F}, \mathcal{F}_f^U)$ to be able to act on $\xi \in \mathcal{F}_g^V$ in the first place, we explicitly use the fact that $\mathcal{F}_g^V$ has $\mathcal{F}$ as a Hilbert space.

We saw in the proof of proposition 19 that $x \otimes \xi \mapsto x(\xi)$ is an isomorphism $\mathcal{F}_f \boxtimes_{\mathcal{A}(S^1_L)} \mathcal{F}_g \to \mathcal{F}_{fg}$. (We specialise to f and g living on the right semicircle so that $U_{f,L}$ and $U_{g,L}$ vanish.) Also, we saw that $x_0 := xU_{f,R}^{-1} \in \mathcal{A}(S^1_L)$.

Now if $U, V \in U(\mathcal{A}_{\mathrm{ev}}(S^1_R))$, then the sector isomorphisms $U : \mathcal{F}_f^U \to \mathcal{F}_f$ and $V : \mathcal{F}_g^V \to \mathcal{F}_g$ identify $\mathcal{F}_f^U \boxtimes \mathcal{F}_g^V$ with $\mathcal{F}_f \boxtimes \mathcal{F}_g$ through $x \otimes \xi \mapsto Ux \otimes V\xi$. Through the isomorphism $\mathcal{F}_f \boxtimes \mathcal{F}_g \to \mathcal{F}_{fg}$, $Ux \otimes V\xi$ is mapped to $Ux(V\xi)$, where $Ux = x_0 U_{f,R}$ with $x_0 \in \mathcal{A}(S^1_R)$. But then

$$\begin{aligned} Ux(V\xi) &= x_0 U_{f,R} V\xi \\ &= x_0(U_{f,R}VU_{f,R}^{-1})U_{f,R}\xi \\ &= (U_{f,R}VU_{f,R}^{-1})x_0 U_{f,R}\xi \\ &= \alpha_f(V)Ux(\xi) . \end{aligned}$$

(We used that $U_{f,R}VU_{f,R}^{-1} \in U(\mathcal{A}_{\mathrm{ev}}(S^1_R))$ commutes with $x_0 \in \mathcal{A}(S^1_L)$.)

This means that $x \otimes \xi \mapsto \alpha_g(V)Ux(\xi)$ is an isomorphism $\mathcal{F}_f^U \boxtimes \mathcal{F}_g^V \to \mathcal{F}_{fg}$. But since $\eta \mapsto \alpha_f(V)U\eta$ defines an isomorphism $\mathcal{F}_{fg}^{\alpha_f(V)U} \to \mathcal{F}_{fg}$, the map $x \otimes \xi \mapsto x(\xi)$ must be an isomorphism $\mathcal{F}_f^U \boxtimes \mathcal{F}_g^V \to \mathcal{F}_{fg}^{\alpha_f(V)U}$.

The class $[\mathcal{F}_{fg}^{\alpha_f(V)U}]$ is independent of the representatives of $[\mathcal{F}_f^U]$ and $[\mathcal{F}_g^V]$; if we replace f and U by βf and $U_\beta U$ and g and V by γg, $U_\gamma U$ (where $\beta, \gamma \in \Omega\mathrm{Spin}(V)$), then the product becomes $\mathcal{F}_{(\beta f \gamma f^{-1})fg}^{\alpha_{\beta f}(U_\gamma V)U_\beta U}$. Now

$$(U_{\beta,R}U_{f,R}U_\gamma U_{f,R}^{-1})U_{f,R}VU_{f,R}^{-1}U_{\beta,R}^{-1}U_\beta = (U_\beta U_{f,R}U_\gamma U_{f,R}^{-1})U_{f,R}VU_{f,R}^{-1}$$

because $U_{\beta,R}^{-1}U_\beta = U_{\overline{\beta}}^{-1} \in \mathcal{A}_{\mathrm{ev}}(S_L^1)$ commutes with $\alpha_f(U_\gamma V)$ and eats the left part of $U_{\beta,R} = U_\beta U_{\overline{\beta}}$. Thus $\mathcal{F}_{\beta f}^{U_\beta U} \boxtimes \mathcal{F}_{\beta g}^{U_\gamma V}$ is brought in isomorphism with the sector $\mathcal{F}_{\mu fg}^{U_\mu \alpha_f(V)U}$, where $\mu = \beta f \gamma f^{-1}$ is a loop. This sector defines the same 2-morphism $[\mathcal{F}_{fg}^{\alpha_f(V)U}]$, so that the map $[\mathcal{F}_f^U] * [\mathcal{F}_g^V] := [\mathcal{F}_{fg}^{\alpha_g(V)U}]$ is defined on 2-morphisms, not just on sectors. This proves the first part of our claim.

The second part is just a variation on this theme. One concludes that the square

$$\begin{array}{ccccccc}
x \otimes \xi & \in & \mathcal{F}_f^U \boxtimes \mathcal{F}_g^V & \longrightarrow & \mathcal{F}_{fg}^{\alpha_f(V)U} & \ni & x(\xi) \\
\big\downarrow & & {\scriptstyle X}\big\downarrow{\scriptstyle Y} & & \big\downarrow {\scriptstyle XU^{-1}\alpha_f(Y)U} & & \big\downarrow \\
Xx \otimes Y\xi & \in & \mathcal{F}_f^{UX^{-1}} \boxtimes \mathcal{F}_g^{VY^{-1}} & \longrightarrow & \mathcal{F}_{fg}^{\alpha_f(VY^{-1})UX^{-1}} & \ni & Xx(Y\xi)
\end{array}$$

is commutative because $\alpha_f(VY^{-1})UX^{-1} = (\alpha_f(V)U)(XU^{-1}\alpha_f(Y)U)^{-1}$ and because, with $x_0 = UxU_{f,R}^{-1} \in \mathcal{A}(S_L^1)$ commuting with the even operator $\alpha_f(Y) \in \mathcal{A}_{\mathrm{ev}}(\mathcal{A}(S_R^1))$, we have

$$\begin{aligned}
Xx(Y\xi) &= XU^{-1}x_0U_{f,R}Y\xi \\
&= XU^{-1}x_0(U_{f,R}YU_{f,R}^{-1})U_{f,R}\xi \\
&= XU^{-1}(U_{f,R}YU_{f,R}^{-1})x_0U_{f,R}\xi \\
&= (XU^{-1}\alpha_f(Y)U)x(\xi)\,.
\end{aligned}$$

The fusion product of $X : \mathcal{F}_f^U \to \mathcal{F}_f^{UX^{-1}}$ and $Y : \mathcal{F}_f^V \to \mathcal{F}_f^{VY^{-1}}$ is thus $X * Y = XU^{-1}\alpha_f(Y)U$. □

We define a strict (and presumably continuous) 2-group that will be weakly equivalent to $G(V)$. Define the semidirect product $L_eO(V) \ltimes PU(\mathcal{A}_{\mathrm{ev}}(S_R^1))$ by

$$(f,U)\cdot(g,V) := (fg, \alpha_f(V)U)\,.$$

Proposition 24 (and definition)**.** *The above semidirect product is indeed a group, and* $\{(\gamma, U_\gamma)\,;\, \gamma \in \Omega\mathrm{Spin}(V)\}$ *is a normal subgroup. We define* Γ_0 *to be the quotient group.*

Proof. The group axioms are just a calcultion:

$$\begin{aligned}(f,U)\big((g,V)(h,W)\big) &= (\, fgh, \alpha_f(\alpha_g(W)V)\,) \\ &= (\, fgh, \alpha_{fg}(W)\alpha_f(V)U\,) \\ &= \big((f,U)(g,V)\big)(h,W)\,,\end{aligned}$$

and the inverse of (f,U) is $(f^{-1}, \alpha_f^{-1}(U^{-1}))$.

Now for $\gamma \in \Omega\mathrm{Spin}(V)$, we have

$$\begin{aligned}(f,U)(\gamma,U_\gamma)(f^{-1},\alpha_f^{-1}(U^{-1})) &= (\, f\gamma f^{-1}, \alpha_{f\gamma}(\alpha_f^{-1}(U^{-1}))\alpha_f(U_\gamma)U\,) \\ &= (f\gamma f^{-1}, U_{f\gamma f^{-1}})\,,\end{aligned}$$

where the last line is valid by the same reasoning as proposition 23. Explicitly, we have

$$\begin{aligned}\alpha_{f\gamma}(\alpha_f^{-1}(U^{-1})\alpha_f(U_\gamma)U &= \big(U_{f,R}U_{\gamma,R}U_{f,R}^{-1}\big)U^{-1}\big(U_{f,R}U_{\gamma,R}^{-1}U_{f,R^{-1}}U_{f,R}U_\gamma U_{f,R}^{-1}\big)U \\ &= \big(U_{f,R}U_{\gamma,R}U_{f,R}^{-1}\big)U^{-1}\alpha_f(U_{\overline{\gamma}})U \\ &= \big(U_{f,R}U_{\gamma,R}U_{f,R}^{-1}\big)\alpha_f(U_{\overline{\gamma}})U^{-1}U \\ &= \big(U_{f,R}U_{\gamma,R}U_{\overline{\gamma}}U_{f,R}^{-1}\big) \\ &= U_{f\gamma f^{-1}}\,,\end{aligned}$$

where we used that $\alpha_f(U_{\overline{\gamma}})$ is in $\mathcal{A}_{\mathrm{ev}}(S^1_L)$ and therefore commutes with $U \in \mathcal{A}(S^1_R)$. The diagonal subgroup $\Omega\mathrm{Spin}(V)$ is thus normal. □

Remark 12. The group Γ_0 is roughly the string group model featured in [ST04].

Now define the group $\Gamma_1 := \{[f,U;X]\,;\,[f,U] \in \Gamma_0\,, X \in \mathcal{A}_{\mathrm{ev}}(S^1_R)\}$ by

$$[f,U;X] * [g,V;Y] := [fg, \alpha_f(V)U; XU^{-1}\alpha_f(Y)U]\,.$$

Note that the term on the right only depends on the class $[f,U]$ modulo $\Omega(\mathcal{A}_{\mathrm{ev}}(S^1_R))$ because for (γ, U_γ) with $\gamma \in \Omega(\mathcal{A}_{\mathrm{ev}}(S^1_R))$, we have (the old song)

$$XU^{-1}U_\gamma^{-1}\alpha_{\gamma f}(Y)U_\gamma U = XU^{-1}U_{\overline{\gamma}}\alpha_f(Y)U_{\overline{\gamma}}^{-1}U = XU^{-1}\alpha_f(Y)U\,.$$

Define the source and target maps $s,t : \Gamma_1 \to \Gamma_0$ by $s([f,U;X]) = [f,U]$ and $t([f,U,X]) = [f,UX^{-1}]$. We give $\Gamma_1 \rightrightarrows \Gamma_0$ a groupoid structure by declaring

$$[f,U,X] \circ [f,UX^{-1},Y] := [f,U,XY]\,.$$

The class of the r.h.s. modulo $\Omega\mathrm{Spin}(V)$ does not depend on the representatives chosen on the l.h.s.

Proposition 25. *With the above multiplication, source and target maps, the groupoid $\Gamma_1 \rightrightarrows \Gamma_0$ is a strict 2-group.*

First of all, $\Gamma_1 \rightrightarrows \Gamma_0$ is really a groupoid, with identity morphisms $[f,U;\mathbf{1}]$. That Γ_1 is a group, is shown by the little calculation

$$\begin{aligned}([f,U;X]*[g,V;Y])*[h,W;Z] &= [fgh,\alpha_f(\alpha_g(W)V)U;XU^{-1}\alpha_f(YV^{-1}\alpha_g(Z)V)U] \\ &= [f,U;X]*([g,V;Y]*[h,W;Z])\,.\end{aligned}$$

The source s is the projection on Γ_0, so certainly a group homomorphism. The target t is a group homomorphism by a second little calculation

$$\begin{aligned}t([f,U,X])*t([g,V,Y]) &= [fg,\alpha_f(V)\alpha_f(Y^{-1})UX^{-1}] \\ &= t([fg,\alpha_f(V)U;XU^{-1}\alpha_f(Y)U])\,.\end{aligned}$$

So $\Gamma_1 \rightrightarrows \Gamma_0$ is a 2-group, i.e. a groupoid with compatible group structures. □

We define the functor $F:\Gamma\to G(V)$. The map on objects, $F_0:\Gamma_0\to G_0(V)$, is given by $F_0([f,U])=[\mathcal{F}_f^U]$, and the map $F_1:\Gamma_0\to G_1(V)$ on the level of morphisms is $F_1([f,U;X])=[\mathcal{F}_f^U]\xrightarrow{X}[\mathcal{F}_f^{UX^{-1}}]$. It is indeed a functor because concatenation is respected,

$$F_1([f,U,X]\circ[f,UX^{-1};Y])=[\mathcal{F}_f^U]\xrightarrow{X}[\mathcal{F}_f^{UX^{-1}}]\xrightarrow{Y}[\mathcal{F}_f^{UX^{-1}Y^{-1}}]\,.$$

This brings us to the main point for now:

Theorem 26. *The functor $F:\Gamma\to G(V)$, together with the natural transformation $N_{f\,g}^{U\,V}:[\mathcal{F}_f^U]*[\mathcal{F}_g^V]\to[\mathcal{F}_{fg}^{\alpha_f(V)U}]$ defined by $x\otimes\xi\mapsto x(\xi)$, and the isomorphism of identity objects $L^2(\mathcal{A}(S^1_+))\to\mathcal{F}:[a]\mapsto a\Omega$, constitutes an isomorphism of weak 2-groups.*

Proof. The functor F is a weak monoidal functor. It respects the left identity isomorphisms because the equivalence $L^2(\mathcal{A}(S^1_+))\boxtimes\mathcal{F}_f^U\to\mathcal{F}_f^U$ in $G(V)$, coming from the equivalence in the fusion category of bimodules, is given by $x\otimes\xi\mapsto x\cdot\xi$, where $x\in\mathrm{Hom}_{-,\mathcal{A}(S^1_L)}(L^2(\mathcal{A}(S^1_+),L^2(\mathcal{A}(S^1_+)))$. Now the isomorphisms $\mathrm{Hom}_{-,\mathcal{A}(S^1_L)}(L^2(\mathcal{A}(S^1_+),L^2(\mathcal{A}(S^1_+)))\simeq\mathcal{A}(S^1_L)$ (that allows x to act on ξ in the fusion category) and the isomorphism $\mathrm{Hom}_{-,\mathcal{A}(S^1_L)}(\mathcal{F},\mathcal{F})\simeq\mathcal{A}(S^1_L)$ (which is needed for the natural transformation $\mathcal{F}\boxtimes\mathcal{F}_f^U\to\mathcal{F}_f^U:x\otimes\xi\mapsto x(\xi)$) are intertwined by the isomorphism of identity objects (cf. prop. 11) $[a]\mapsto a\Omega$. Indeed, if $x:\mathcal{F}\to\mathcal{F}$ is in $\mathcal{A}(S^1_+)\cap\mathcal{A}(S^1_L)$, then the induced operator $L^2(\mathcal{A}(S^1_+))\to L^2(\mathcal{A}(S^1_+))$ is simply $[a]\mapsto[xa]$, whereas for $x\in\kappa^{-1}\mathcal{A}(S^1_-)\kappa\cap\mathcal{A}(S^1_L)$, it is $[a]\mapsto[Jx^\dagger Ja]$. This is $[\Lambda(F)(\kappa x\kappa^{-1})\Lambda(F)]$, which is the action of $\kappa x\kappa^{-1}\in\mathcal{A}(S^1_-)$ on $L^2(S^1_+)$ according to the definition of the vacuum sector. The fact that F respects the right identity is proven the same way.

We need to show that F and N preserve the associators, i.e. that the diagram

$$\begin{array}{ccccc}(\mathcal{F}_f^U\boxtimes\mathcal{F}_g^V)\boxtimes\mathcal{F}_h^W & \xrightarrow{N\otimes\mathbf{1}} & \mathcal{F}_{fg}^{\alpha_f(V)U}\boxtimes\mathcal{F}_h^W & \xrightarrow{N} & \mathcal{F}_{fgh}^{\alpha_{fg}(W)\alpha_f(V)U} \\ \downarrow{\scriptstyle\alpha_{f\,g\,h}^{U\,V\,W}} & & & & \Vert \\ \mathcal{F}_f^U\boxtimes(\mathcal{F}_g^V\boxtimes\mathcal{F}_h^W) & \xrightarrow{\mathbf{1}\otimes N} & \mathcal{F}_f^U\boxtimes\mathcal{F}_{gh}^{\alpha_g(W)V} & \xrightarrow{N} & \mathcal{F}_{fgh}^{\alpha_f(\alpha_g(W)V)U}\end{array}$$

commutes. Now the associator $(\alpha_{f\,g\,h}^{U\,V\,W})^{-1}$ for the Connes fusion products identifies

$$x \otimes (y \otimes \xi) \in \mathrm{Hom}_{-,\mathcal{A}(S^1_L)}(\mathcal{F}, \mathcal{F}^U_f) \otimes (\mathrm{Hom}_{-,\mathcal{A}(S^1_L)}(\mathcal{F}, \mathcal{F}^V_g) \otimes \mathcal{F}^W_h)$$

with

$$(\eta \mapsto x \otimes y(\eta)) \otimes \xi \in \mathrm{Hom}_{-,\mathcal{A}(S^1_L)}(\mathcal{F}, \mathrm{Hom}_{-,\mathcal{A}(S^1_L)}(\mathcal{F}, \mathcal{F}^U_f) \otimes \mathcal{F}^V_g) \otimes \mathcal{F}^W_h)\,.$$

We simply follow the heptagon.

$$\begin{array}{ccccc} \big(\eta \mapsto x \otimes y(\eta)\big) \otimes \xi & \xrightarrow{N \otimes \mathbf{1}} & \big(\eta \mapsto x(y(\eta))\big) \otimes \xi & \xrightarrow{N} & x(y(\xi)) \\ \big\downarrow \alpha_{f\,g\,h}^{U\,V\,W} & & & & \big\| \\ x \otimes (y \otimes \xi) & \xrightarrow{\mathbf{1} \otimes N} & x \otimes y(\xi) & \xrightarrow{N} & x(y(\xi)) \end{array}$$

This shows that F and N constitute a monoidal functor.

We show that the monoidal functor F is essentially surjective. By proposition 21, every D_g-D_1 sector is isomorphic to either $\mathcal{F}_g$ or its shifted version. According to propositions 22 and 13, the operator U_ω, where ω is a loop in $\Omega SO(V)$ with odd winding number, flips the grading, so that the D_g-D_1 modules $\mathcal{F}_{\omega g}$ and $\mathcal{F}_g$ are not equivalent. This means that $\mathcal{F}_g$ is isomorphic to the image of $[g, \mathbf{1}]$ under F, and its twisted version is isomorphic to the image of $[\omega g, \mathbf{1}]$. (It has the same endpoint and is not isomorphic to $\mathcal{F}_g$.)

The functor F is full because according to prop. 22, there are isomorphisms $X : [\mathcal{F}^U_f] \to [\mathcal{F}^V_g]$ if and only if $\gamma := gf^{-1} \in \Omega\mathrm{Spin}(V)$, in which case $X = V^{-1}U_\gamma U$, where U runs over the representatives of $\overline{U} \in PU(\mathcal{A}_{\mathrm{ev}}(S^1_L))$. (So $\mathrm{Hom}([\mathcal{F}^U_f], [\mathcal{F}^V_g]) \simeq U(1)$.) Because in that case $[\mathcal{F}^V_g] = [\mathcal{F}^{U_\gamma UX^{-1}}_{\gamma f}] = [\mathcal{F}^{UX^{-1}}_f]$, we have X as the image $F([f, U, X])$. Conversely, F is faithful because if $F([f, U, X]) = F([g, V, Y])$, then $X : \mathcal{F}^U_f \to \mathcal{F}^{UX^{-1}}_f$ defines the same morphism $[\mathcal{F}^U_f] \to [\mathcal{F}^{UX^{-1}}_f]$ as $Y : \mathcal{F}^V_g \to \mathcal{F}^{VY^{-1}}_g$. In particular, $g = \gamma f$ and $V = U_\gamma U$ (prop. 22), and $X = Y$. □

6 Spin structures and 2-group bundles

We construct the analogue of a Spin^c-structure and of the spinor bundle on loop space LM, and show that its transformation behaviour is governed by the string 2-group. The analogue of the action of the bundle of Clifford algebras will be the action of a bundle of defects over LM. We make some tentative remarks on fusion structures on these bundles.

6.1 Heuristics

The inspiration for this chapter is the following heuristic account. Let (M, g) be a Riemannian manifold. Consider the 3-category Path where the objects are smooth paths $p\colon [0,\pi] \to M$. There are 1-morphisms between p and p' only if $p(\pi) = p'(0)$, in which case a 1-morphism is an isometry $g \in \mathrm{SO}(T_{p(\pi)}M, g)$. If $g\colon p \to p'$ and $g'\colon p' \to p$, then a 2-morphism $F\colon g \to g'$ is a framing of the continuous loop $p' \circ p\colon S^1 \to M$, i.e. a lift $F\colon S^1 \to FM$ with the property that F is smooth outside $\{0,\pi\}$, and that the discontinuities at 0 and π are given by g' and g respectively. Finally, a 3-morphism between framings F and F' is a lift of $F^{-1}F' \in LSO(\mathbb{R}^d)$ to $\widehat{LSO(\mathbb{R}^d)}$.

There should be a functor from this path 3-category (or rather, thing of which I am too lazy to check whether or not it is a 3-category) to the 3-category (thing that André claims without proof that it is a 3-category) of conformal nets.

It should map an object $p\colon [0,\pi] \to M$ to the free fermionic theory living on $\mathrm{Im}(p)$, a 1-morphism $g\colon p \to p'$ to a defect isomorphic to D_g, a 2-morphism $F\colon g \to g'$ to a sector $\mathcal{F}$ canonically constructed from the framing. This is the thing you actually construct, it is isomorphic to $\mathcal{F}_{g,g'}$. You then prove that the defects and nets that arise in this way do not depend on the choice of framing, so you have definded the functor on objects, 1-morphisms and 2-morphisms in a single stroke. A 3-morphism $\hat{f}$ such that f is the difference between two framings gets sent to the unitary intertwiner $U_{\hat{f}}$ of sectors.

Rather than trying to make this precise (which should be considerably easier than proving that conformal nets form a 3-category, because all that is needed is the free fermionic net), we focus on the 2-group picture.

6.2 Spin^c-structures

First, we review the construction of a Spin^c-structure on a smooth manifold with compatible Riemannian and almost complex structures.

Let L be a Fréchet manifold with smoothly varying positive definite bilinear form $G\colon TL \times TL \to \mathbb{R} \times L$, and let $T^{\mathrm{cl}}_\phi L$ be the real Hilbert completion w.r.t. G_ϕ of the tangent space $T_\phi L$ at $\phi \in L$. Let $J\colon T^{\mathrm{cl}}L \to T^{\mathrm{cl}}L$ be a smooth almost complex structure that respects G, i.e., $J_\phi^2 = -1$ and $G_\phi(J_\phi v, J_\phi w) = G_\phi(v,w)$ for all $v, w \in T_\phi L$. Since $(J+i)(J-i) = 0$, we can split $T^{\mathrm{cl}}_\phi L \otimes_{\mathbb{R}} \mathbb{C} = T^{1,0}_\phi L \oplus T^{0,1}_\phi L$ as a direct sum with $T^{1,0}_\phi L := \mathrm{Ker}(J_\phi - i)$ and $T^{0,1}_\phi L := \mathrm{Ker}(J_\phi + i)$. This direct sum is orthogonal w.r.t. the hermitean inner product $\langle v, w\rangle = g_\phi(\overline{v}, w)_{\mathbb{C}}$. Indeed, we have $\overline{T^{1,0}L_\phi} = T^{0,1}_\phi L$ and vice versa, because J, as the complexification of a real operator, respects complex conjugation. In detail: for $v \in T^{1,0}_\phi L$, we thus have $J\overline{v} = \overline{Jv} = \overline{iv} = -i\overline{v}$ and hence $\overline{v} \in T^{0,1}_\phi L$. Now $T^{1,0}_\phi L \perp T^{0,1}_\phi L$ follows from the fact that for $v \in T^{1,0}_\phi L$, $w \in T^{0,1}_\phi L$, we have

$$\langle v, w\rangle = G_\phi(\overline{v}, w)_{\mathbb{C}} = G_\phi(J\overline{v}, Jw)_{\mathbb{C}} = G_\phi(-i\overline{v}, -iw)_{\mathbb{C}} = -\langle v, w\rangle\,.$$

If the map $L \to B(T^{\mathrm{cl}}L_{\mathbb{C}})\colon \phi \mapsto \mathbf{1}_{\mathbb{R}_{\geq 0}}(J_\phi)$ varies continuously on L, then we obtain a continuous polarisation $T^{\mathrm{cl}}_{\mathbb{C}}L = T^{1,0}L \oplus T^{0,1}L$ of the complexified closure of the tangent bundle of L.

Using only the metric G, we can construct the smooth bundle of Clifford algebras $\mathrm{Cl}_\phi := \mathrm{Cl}(T^{\mathrm{cl}}_\phi L \otimes_{\mathbb{R}} \mathbb{C}, G_\phi)$ as the universal algebra with relations $\psi \cdot \psi' + \psi' \cdot \psi = G_\phi(\psi, \psi')$ for the generators $\psi, \psi' \in T_\phi L \otimes_{\mathbb{R}} \mathbb{C}$.

Using the complex structure J, we can construct in every point $\phi \in L$ the Cl_ϕ representation $\mathcal{F}^0_\phi := \mathrm{Cl}_\phi / \mathrm{Cl}_\phi T^{0,1}_\phi L \simeq \wedge T^{1,0}_\phi L$. The closure $\mathcal{F}_\phi$ of $\mathcal{F}^0_\phi$ with respect to the hermitean inner product of $\wedge T^{1,0}_\phi L$ carries a $*$-representation of Cl_ϕ, where the adjoint is implemented by $(\psi_1 \ldots \psi_n)^* = \overline{\psi}_n \cdots \overline{\psi}_1$. We thus obtain for every $\phi \in L$ a von Neumann algebra M_ϕ, namely the closure of Cl_ϕ with respect to the strong topology on $\mathcal{F}_\phi$. Equivalently, M_ϕ is the closure of Cl_ϕ w.r.t. the state $a \mapsto \langle \Omega_\phi, a\Omega_\phi \rangle$, where $\Omega_\phi := [1] \in \mathcal{F}_\phi$ is the canonical vacuum vector in the Fock space $\mathcal{F}_\phi$. The GNS-representation of Cl_ϕ with respect to this state is of course precisely $\mathcal{F}_\phi$.

We would like to say that $\mathcal{F} \to L$ is a vector bundle of Cl_ϕ-representations. If $U \subset L$ and $U' \subset L$ are open sets over which TL is trivialised, and $F \colon U \times W \xrightarrow{\sim} TL|_U$ and $F' \colon U' \times W \xrightarrow{\sim} TL|_{U'}$ are isometric trivialisations, then for $\phi \in U \cap U'$, the transition $F^{-1}_\phi F'_\phi \colon W \to W$ induces an automorphism of $\mathrm{Cl}(W)$. For $\mathcal{F} \to L$ to constitute a bundle of Clifford representations, it is necessary and sufficient that the transition functions $F^{-1}F \colon U \cap U' \to B(W)$ lift to unitary operators on $\mathcal{F}_\phi$. (The cocycle condition on the overlap of 3 trivialisations is then automatically satisfied.) By Segal's quantisation criterion, this is equivalent to the subset $\{F_\phi \colon W \to T^{\mathrm{cl}}_\phi L\,;\, F^* G_\phi = G_0\,, \mathrm{Tr}|J_\phi F_\phi - F_\phi J_0|^2 < \infty\}$ of OFL to be a $O_{\mathrm{res}}(W, W^{1,0} \oplus W^{0,1})$-subbundle, where the restricted orthogonal group is the group of orthogonal transformations $O \colon W \to W$ such that $[O, P_{W^{1,0}}]$ is Hilbert-Schmidt, and (W, G_0, J_0) is the local model for the tangent space.

If L is a riemannian manifold of finite dimension $2k$ with compatible smooth complex structure J, then the O_{res}-condition is automatically fulfilled. The above construction then produces a smooth Spin^c-structure $Q \to OF(L)$ on L as follows. Given an orthogonal frame $f \in OF_\phi L$, let α_f be the induced isomorphism $\alpha_f : \mathrm{Cl}_0 \to \mathrm{Cl}_\phi$, where $\mathrm{Cl}_0 := \mathrm{Cl}(\mathbb{R}^k \oplus \mathbb{R}^k \otimes_{\mathbb{R}} \mathbb{C})$. Construct the 'standard' Clifford representation $\mathcal{F}_0 := \mathrm{Cl}_0 / \mathrm{Cl}_0 \cdot V_0^- \simeq \wedge V_0^+$ with $V_0^\pm := \{v \oplus (\pm iv)\,;\, v \in \mathbb{R}^k \otimes_{\mathbb{R}} \mathbb{C}\}$, and set $Q_f := \{u \colon \mathcal{F}_0 \to \mathcal{F}_\phi\,;\, \alpha_f(a) = uau^{-1}\}$ to be the set of unitary frames of $\mathcal{F} \to L$ that implement α_f. Then $Q_\phi = \cup_{f \in OF_\phi(L)} Q_f$ is a principal fibre bundle over L with structure group Spin^c, and $Q \to OF(L)$ covers the group homomorphism $\mathrm{Spin}^c \to \mathrm{SO}(\mathbb{R}^{2k})$.

6.3 The frame 2-bundle on Loop space

For loop spaces, the situation is somewhat more involved, because although there is a canonical complex structure on the tangent spaces of the loop space at every individual loop, its reluctance to vary smoothly renders the above construction invalid, forcing us to introduce string groups and their associated structures instead.

We start by describing loop space and a sub-2-bundle of its frame bundle. Roughly speaking, this higher structure serves to divide the set of pairs of frames into ones which are connected by 'inner' transformations (in O_{res}) and ones that are not.

6.3.1 Coordinates on loop space

Let M be a d-dimensional connected manifold. We define

$$PM := \{(p_L, p_R) \in C^\infty([0,\pi], M) \times C^\infty([\pi, 2\pi], M)\,;\, p_L(\pi) = p_R(\pi)\}$$

to be the Fréchet manifold of continuous paths which are smooth outside π, and whose left and right derivatives of all orders extend continously to 0, π and 2π. We will identify (p_L, p_R) with the continuous function $p\colon [0, 2\pi] \to M$ that is smooth outside π. Let $LM := \{\phi \in PM\,;\, \phi(0) = \phi(2\pi)\}$ be the submanifold of closed paths. If $x \in M$ is a distinguished point, then $P_x M := \{p \in PM\,;\, p(0) = x\}$ and $\Omega M := \{\phi \in LM\,;\, \phi(0) = x\}$.

Note that $L\mathbb{R}^d$, as the kernel of the continuous map

$$C^\infty([0,\pi], \mathbb{R}^d) \times C^\infty([\pi, 2\pi], \mathbb{R}^d) \to \mathbb{R}^d \quad : \quad (f, f') \mapsto f'(\pi) - f(\pi)$$

of Fréchet spaces, is a Fréchet space itself. Suppose that (M, g) is Riemannian. (If it is not, we endow it with a metric.) Using the Levi-Civita connection ∇ on M, the connected component LM_0 containing the constant loops $M \hookrightarrow LM$ can be modelled on $L\mathbb{R}^d$ as follows.

A *framing* of $\phi \in LM$ is an element $F \in P(OFM)$ such that $\pi \circ F = \phi$. We will call a framing *closed* if $F(2\pi) = F(0)$. Given a loop $\phi_0 \in LM_0$ and a closed framing $F^0 \in L(OFM)$ of ϕ_0, we construct a chart $\kappa_{F^0} : L(B^d_\varepsilon(0)) \to LM_0$ by $(\kappa_{F^0} v)(\theta) := \exp^{\nabla}_{\phi_0(\theta)}(F^0_\theta(v(\theta)))$, the geodesic flow from $\phi_0(\theta)$ in the direction $F^0_\theta(v(\theta)) \in T_{\phi_0(\theta)} M$. The transition functions take values in $L(\mathbb{R}^d \rtimes O(\mathbb{R}^d))$; they act by affine (and therefore smooth) transformations on $L\mathbb{R}^d$, which shows that LM_0 is a smooth manifold. For the other connected components, a similar trick goes through[1].

Since the transition functions live in in $L(\mathbb{R}^d \rtimes O(\mathbb{R}^d))$, the frame bundle $FLM \to LM$ has a principal subbundle with structure group $L(O(\mathbb{R}^d)) < GL(L(\mathbb{R}^d))$, with affine transition functions[2].

For simplicity, we will assume from now on that M is oriented, and all transformations, frames, etc. are orientation preserving. Accordingly, we will denote by OFM the principal $\mathrm{SO}(\mathbb{R}^d)$-bundle of orientation preserving orthogonal frames.

[1] The only difference being that if M is not orientable, the local model for the orientation flipping loops is two copies of $C^\infty([0,\pi], \mathbb{R}^d)$ glued together by an element of $O(\mathbb{R}^d)$ with determinant -1. For d even, this gives rise to twisted loop algebras, cf. Rk. 3 on page 14.

[2] Note that $L(\mathbb{R}^d \rtimes O(\mathbb{R}^d))$ has a projective unitary representation (by bosonic field operators with partcle number ≤ 2) on the Hilbert closure of the bosonic Fock space. This probably accounts for the 'horizontal' part of a typical fibre $L^2(C^\infty(S^1, \mathbb{R}^{d-1}), \mu_G) \otimes \mathcal{F}_0 \simeq \mathcal{F}_B \otimes \overline{\mathcal{F}}_B \otimes \mathcal{F}$, where the bosonic fields should act as infinitesimal translations. I would like to understand this better.

The tangent space $TLM \to LM$ has fibre $T_\phi LM = \Gamma(\phi^*TM) := \{v \in LTM\,;\, \pi \circ v = \phi\}$, so that $TLM \to LM$ is precisely $LTM \to LM$. We introduce the inner product

$$G_\phi(s, s') := \frac{1}{2\pi}\int_0^{2\pi} g_{\phi(\theta)}(s(\theta), s'(\theta))d\theta,$$

which is smooth and nondegenerate.

Remark 13. It is tempting to add a factor $\|\phi'(\theta)\|_g$ in the above integral, and consider the L^2 vector fields on $\phi(S^1)$ rather than the L^2 sections of ϕ^*TM. I will refrain from this for now, because of the trouble this causes when ϕ is not injective (the Hilbert space vanishes for constant loops, etc.). Then again, if one aims at some sort of localisation mechanism at the constant loops and crossings, this might be just the sort of trouble one is looking for.

A framing F of $\phi \in LM$ is determined by its initial frame $F(0) \in OF_{\phi(0)}M$ and the unique connection ∇_F on $\phi^*TM \to S^1$ with respect to which F is covariantly constant. Denote by $\mathrm{hol}(F) := F(2\pi)F(0)^{-1} \in \mathrm{SO}(T_{\phi(0)}M)$ the holonomy of ∇_F, and by $h(F) := F(0)^{-1}F(2\pi) = F(0)^{-1} \circ \mathrm{hol}(F) \circ F(0)$ its pullback to $\mathrm{SO}(\mathbb{R}^n)$. Under the $\mathrm{SO}(\mathbb{R}^d)$-action $F \mapsto F \circ g$, $\mathrm{hol}(F)$ is invariant and $h = F(0)^{-1}F(2\pi)$ transforms under the adjoint action.

Let $L_h\mathbb{R}^d$ be the Fréchet space of continuous functions $v\colon \mathbb{R} \to \mathbb{R}^d$ such that v is smooth outside $\mathbb{Z}\pi$, its left and right derivatives of all orders extend continuously to $[n\pi, (n+1)\pi]$, and $v(\theta+2\pi) = h^{-1}v(\theta)$. It carries the continuous inner product $(v, w) := \frac{1}{2\pi}\int_0^{2\pi}(v(\theta), w(\theta))_0 d\theta$, with restpect to which, of course, $L_h\mathbb{R}^d$ is not complete.

For every framing F of $\phi \in LM$, we define the isomorphism $K_F\colon L_h\mathbb{R}^d \to \Gamma(\phi^*TM)$ by $(K_F v)(\theta) := F_\theta(v(\theta))$. To check that K_F is well defined, we extend $F\colon [0, 2\pi] \to OFM$ to a continuous map $F\colon \mathbb{R} \to OFM$ by $F(\theta + 2\pi) = F(\theta)h$, and note that $F(\theta + 2\pi)(v(\theta + 2\pi)) = F(\theta)(h(h^{-1}v(\theta))) = F(\theta)(v(\theta))$. It is an isometry because the frames are orthogonal.

This allows for an explicit description of the local trivialisation $\kappa_{F^0\,*}\colon L(B_\varepsilon(0)) \times L\mathbb{R}^d \to TLM|_{\mathrm{Im}(\kappa_F)}$. If $\phi(\theta) = \exp^{\nabla}_{\phi_0(\theta)}(v(\theta))$ is in the image of κ_{F^0}, then the closed framing F_0 of ϕ_0 is pushed forward along the geodesic flow to a closed framing F_ϕ of ϕ, and the fibre of $\kappa_{F^0\,*}\colon TL\mathbb{R}^d \to TLM$ over ϕ is precisely $K_{F_\phi}\colon L_e\mathbb{R}^d \to \Gamma(\phi^*TM)$.

Every orthogonal framing of ϕ introduces a continuous $\mathbb{R}$-action on the Fréchet space $T_\phi LM$ by $(T_t s)(\theta) = F(\theta)F^{-1}(\theta - t)s(\theta - t)$. (Again, we consider the framing as a continuous function $F\colon \mathbb{R} \to OFM$ with $F(\theta + 2\pi) = F(\theta)h$.) This is the parallel transport along the connection on ϕ^*TM for which F is constant. If the framing is closed (or, more generally, if h is of finite order), then T_t factors through an S^1-action.

In turn, the $\mathbb{R}$-action induces a polarisation $T^{\mathrm{cl}}_\phi LM_{\mathbb{C}} = T^{1,0}_\phi LM \oplus T^{0,1}_\phi LM$, where $T^{1,0}_\phi LM$ contains the image of the spectral projection on $\mathrm{Spec}(-i\frac{d}{dt}|_0 T_t) \cap \mathbb{R}^{>0}$ and $T^{0,1}_\phi LM$ the image of the projection on $\mathrm{Spec}(-i\frac{d}{dt}|_0 T_t) \cap \mathbb{R}^{<0}$. Since the pullback along K_F of the $\mathbb{R}$-action is the ordinary shift $T_t v(\theta) = v(\theta - t)$,

$$\begin{aligned}(K_F^{-1} T_t K_F v) &= (K_F^{-1} T_t)(\theta \mapsto F_\theta(v(\theta))) \\ &= K_F^{-1}(\theta \mapsto F_\theta(v(\theta - t))) \\ &= \theta \mapsto v(\theta - t)\,,\end{aligned}$$

and since the closure of $L^h(\mathbb{R}^d)_\mathbb{C}$ is simply $L^2(S^1, \mathbb{C}^d)$, the spectrum of $D := -i\frac{d}{dt}|_0 T_t$ is $\mathbb{Z}$. Now $T^{1,0}_\phi LM_\mathbb{C}$ contains the positive part $K_F\, \overline{\mathrm{span}}(e^{ik\theta}v\,;\, k > 0, v \in \mathbb{C}^n)$ and likewise $T^{0,1}_\phi LM_\mathbb{C}$ the negative part, but we have yet to decide what to do with the kernel of D.

One option is just to include it in $T^{1,0}_\phi LM_\mathbb{C}$, but then the corresponding almost complex structure J would not be real. We therefore split the kernel in two equal parts, to which end we assume that

M is of even dimension $d = 2k$.

We then split $\mathbb{C}^d = \mathbb{C}^k_{1,0} \oplus \mathbb{C}^k_{0,1}$, and add $K_F \mathbb{C}^k_{1,0}$ to $T^{1,0}M$ and $K_F \mathbb{C}^k_{0,1}$ to $T^{0,1}M$. Note that for generic h, none of the eigenvectors $K_F(\theta \mapsto e^{-k\theta})$ belong to $T_\phi LM_\mathbb{C} \subset T^{\mathrm{cl}} LM_\mathbb{C}$.

The framings coming from a trivialisation are always closed. Two such framings, belonging to the same loop ϕ in different charts, differ by a loop $g \in L\mathrm{O}(\mathbb{R}^d)$.

But there is also a second class of relevant framings, namely the ones that are covariantly constant along ϕ for the Levi-Civita connection ∇. Two such framings of the same loop differ by a *constant* $g \in \mathrm{SO}(\mathbb{R}^d)$. Parallel transport along the Levi-Civita connection induces an $\mathbb{R}$-action on $T^{\mathrm{cl}}LM$ by $(r_t s)(\theta) := \exp^{\phi,\nabla}_{\theta,\theta-t} s(\theta - t)$, where $\exp^{\phi,\nabla}_{\theta,\theta-t} : T_{\phi(\theta-t)}M \to T_{\phi(\theta)}M$ is the parallel transport along $\phi|_{[\theta-t,\theta]}$ w.r.t. ∇.

(Note that $r\colon \mathbb{R} \times T^{\mathrm{cl}}LM \to T^{\mathrm{cl}}LM$ is an action by continuous vertical automorphisms of $T^{\mathrm{cl}}LM \to LM$ that respects the metric G – in contrast to the pushforward of the canonical $\mathbb{T}^1$-action $\tilde{r}_t\phi(\theta) = \phi(\theta - t)$, which does not respect the fibre map.)

The connection on M thus singles out – for each initial frame $f_0 \in OF_{\phi(0)}M$ – a distinguished polarisation $T^{\mathrm{cl}}_\phi LM_\mathbb{C} = T^{1,0}_\phi LM \oplus T^{1,0}_\phi LM$. We define the unbounded operator $D := -i\frac{d}{dt}|_0 r_t$ on $T^{\mathrm{cl}}_\phi L_\mathbb{C}$, and set $J_{\phi,f_0} := iP(T^{1,0}_\phi LM) - iP(T^{0,1}_\phi LM)$.

We have seen that every framing of ϕ induces an isomorphism $L^2(S^1, \mathbb{C}^d) \to T^{\mathrm{cl}}LM$ which intertwines the standard polarisation $L^2(S^1, \mathbb{C}^d) = V^0_+ \oplus V^0_-$ with the polarisation of $T^{\mathrm{cl}}LM$ induced by F. Every two framings F and F' differ by an element $F^{-1}F' = g \in PO(\mathbb{R}^d)$. If F and F' both come from a chart of LM, then they are closed and $g \in LO(\mathbb{R}^d)$. According to Segal's quantisation criterion, their polarisations ore then in the same class, i.e., they agree up to Hilbert-Schmidt operators.

If, on the other hand, F comes from a chart and F' is constant w.r.t. the Levi-Civita connection, then $g \in PO(\mathbb{R}^d)$ is in general not closed; $g^{-1}(2\pi)g(0) =$

h, the holonomy of ∇ along ϕ pulled back to $\mathbb{R}^d$. Unless (M, g) is flat, this means that there exists a loop ϕ for which g has a discontinuity, and that the corresponding isomorphism $L^2(S^1, \mathbb{C}^d) \to L^2(S^1, \mathbb{C}^d)\colon v \mapsto gv$ does not preserve the polarisation class.

This tension between framings that respect the smooth structure of LM on the one hand, and on the other hand the framings that faithfully represent the natural polarisation induced by the Levi-Civita connection, is something we have to deal with if we intend to do justice to the Clifford action on the spinor bundle over LM.

6.4 The 2-bundle of framings

We introduce the 2-bundle of framings, an object that controls in a systematic way all the framings, i.e. $L(\mathbb{R})$-linear isomorphisms $F\colon L^h(\mathbb{R}^d) \xrightarrow{\sim} T_\phi LM$, that we need. The relations between them are governed by the 2-group $P\mathrm{SO}(\mathbb{R}^d) \ltimes \Omega\mathrm{Spin} \rightrightarrows PSO(\mathbb{R}^d)$, the cover of which is the string group model of [BCSS07].

6.4.1 The string 2-group

Let $\Gamma_1 \rightrightarrows \Gamma_0$ be the smooth 2-group defined by $\Gamma_0 := P\mathrm{SO}(\mathbb{R}^d)$ and $\Gamma_1 := \Gamma_0 \ltimes L\mathrm{SO}(\mathbb{R}^d)_+$, with $L\mathrm{SO}(\mathbb{R}^d)_+$ the space of loops $g \in \mathrm{SO}(\mathbb{R}^d)$ with $[g] \in \pi_1(\mathrm{SO}(\mathbb{R}^d), g(0))$ even[3]. The target map is $t\colon (p, f) \mapsto p$, the source map is $s\colon (p, f) \mapsto pf$, and the product is the the pointwise product $g * g'(\theta) = g(\theta)g'(\theta)$ on Γ_0 and the semidirect product $(g, f) * (g', f') = (gg', \alpha_{g'}^{-1}(f)f')$ on Γ_1, where $\alpha\colon P\mathrm{SO}(\mathbb{R}^d) \to \mathrm{Aut}(L\mathrm{SO}(\mathbb{R}^d)_+)$ is conjugation, $\alpha_g(f) := gfg^{-1}$.

Evaluation in zero is a morphism $\mathrm{ev}_0\colon \Gamma \to \mathrm{SO}(\mathbb{R}^d)//\mathrm{SO}(\mathbb{R}^d)$ onto the action groupoid of rightmultiplication by $\mathrm{SO}(\mathbb{R}^d)$ on itself. The latter is a 2-group, equipped with the obvious group structure on $\mathrm{Ob}(\mathrm{SO}(\mathbb{R}^d)//\mathrm{SO}(\mathbb{R}^d)) = \mathrm{SO}(\mathbb{R}^d)$, and with the semidirect product $(g, h) * (g', h') = (gg', g'^{-1}gg'h')$ on $\mathrm{Mor}(\mathrm{SO}(\mathbb{R}^d)//\mathrm{SO}(\mathbb{R}^d)) = \mathrm{SO}(\mathbb{R}^d) \ltimes \mathrm{SO}(\mathbb{R}^d)$. Let K be its kernel. Because $P_e\mathrm{Spin} = P_e\mathrm{SO}(\mathbb{R}^d)$ and $\Omega\mathrm{SO}(\mathbb{R}^d)_+ = \Omega\mathrm{Spin}$, we have $K_0 = P_e\mathrm{Spin}$ and $K_1 = P_e\mathrm{Spin} \ltimes \Omega\mathrm{Spin}$. As the kernel of the morphism ev_0 of 2-groups, K is a normal 2-group. Because the image $\mathrm{SO}(\mathbb{R}^d)//\mathrm{SO}(\mathbb{R}^d)$ of ev_0 sits inside Γ as the 2-group of constant paths, we write $\Gamma = (\mathrm{SO}(\mathbb{R}^d)//\mathrm{SO}(\mathbb{R}^d)) \ltimes K$, where the product is twisted by the adjoint action.

Now since $\mathrm{ev}_{2\pi}\colon K \to \mathrm{Spin}_{\mathrm{dis}}$, the evaluation in 2π, is a strict morphism with kernel K' equal to $\Omega\mathrm{Spin} \ltimes \Omega\mathrm{Spin} \rightrightarrows \Omega\mathrm{Spin}$, the action groupoid of the right action of $\Omega\mathrm{Spin}$ on itself, with the 2-group structure coming from the adjoint action. We have the extension

$$1 \to K' \longrightarrow K \xrightarrow{\mathrm{ev}_{2\pi}} \mathrm{Spin}_{\mathrm{dis}} \to 1$$

which, unlike the morphism ev_0 on Γ, does not split. Because K_0 is contractible, the homotopy groups of K'_0 are those of Spin, shifted up by one degree.

[3]For $d > 2$, this is just the connected component $L\mathrm{SO}(\mathbb{R}^d)_0$.

Note that K admits central extensions

$$1 \to (U(1) \rightrightarrows *) \to \hat{K}_c \to K \to 1\,,$$

where $\hat{K}_c$ is the string 2-group $P_e\mathrm{Spin} \ltimes \widehat{\Omega\mathrm{Spin}} \rightrightarrows P_e\mathrm{Spin}$ of [BCSS07] at level $c \in \mathbb{Z}_+$. We will be interested in the level prescribed by the fermionic representation of $\widehat{\Omega\mathrm{Spin}}$ as described in section 3.2. The level there is expressed in terms of the Killing form, which is $1/(d-2)$ times the trace pairing. We convert this to the usual definition of 'level', w.r.t. the invariant bilinear form in which long roots square to 2. This (hopefully) comes down to the trace pairing $(x, y) = -\mathrm{tr}(xy)$ for the simply laced orthogonal algebras $\mathfrak{so}(3) = A_1$, $\mathfrak{so}(6) = A_3$ and $\mathfrak{so}(2l) = D_l$ for $l \geq 4$, and to one half times the trace pairing for the non-simply laced algebras $\mathfrak{so}(2l+1) = B_l$ for $l \geq 2$. For $\mathfrak{so}(4) = A_1 \times A_1$, we hence get twice the trace pairing. This leads to $c = 1$ for the simply laced case, $c = 2$ for the non-simply laced case, and $c = \frac{1}{2} \oplus \frac{1}{2}$ for $\mathfrak{so}(4)$. I'll have to check this more carefully though.

The central extension is constructed most naturally by the fermionic representation of $L\mathrm{SO}(\mathbb{R}^d) > \Omega\mathrm{Spin}$, so we even have a 'string cover' $q\colon \hat{\Gamma} \to \Gamma$. At least if d is even, one can arrange the polarisation of $L^2(S^1, \mathbb{R}^d)_{\mathbb{C}}$ used in the construction to coincide with a polarisation of the constant loops $\mathbb{C}^d$, so that the restriction of q to the preimage of the constant paths $\mathrm{SO}(\mathbb{R}^d)_{\mathrm{dis}}$ is precisely the Spin^c cover;

$$\begin{array}{ccc} \mathrm{SO}(\mathbb{R}^d)/\!/\mathrm{Spin}^c & \longrightarrow & \hat{\Gamma} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q} \\ \mathrm{SO}(\mathbb{R}^d)/\!/\mathrm{SO}(\mathbb{R}^d) & \to & \Gamma \end{array}\,.$$

6.4.2 The frame 2-bundle

Let (M, g) be a Riemannian manifold. Let P be the groupoid $P_1 \rightrightarrows P_0$ with $P_0 := \{F \in POFM\,;\, \pi \circ F \in LM\}$ and $P_1 := P_0 \times L\mathrm{SO}(\mathbb{R}^d)_+$, with $L\mathrm{SO}(\mathbb{R}^d)_+$ the space of loops $f \in \mathrm{SO}(\mathbb{R}^d)$ with $[f] \in \pi_1(\mathrm{SO}(\mathbb{R}^d), f(0))$ even. The target is $t(F, f) = F$ and the source is $s(F, f) = F \circ f$, where $(p \circ f)(\theta)$ is the composition of $f(\theta) \in \mathrm{SO}(\mathbb{R}^d)$ with the frame $F(\theta) : \mathbb{R}^d \to T_{\phi(\theta)}M$.

Let $r\colon \mathrm{ev}^*OFM \to LM$ be the pullback of $OFM \to M$ under the evaluation $\mathrm{ev}_0\colon LM \to M$, and let $\mathrm{ev}^*OFM/\!/\mathrm{SO}(\mathbb{R}^d)$ be the corresponding action groupoid, a $\mathrm{SO}(\mathbb{R}^d)/\!/\mathrm{SO}(\mathbb{R}^d)$-2-bundle over LM.

We have a morphism $P \to \mathrm{ev}^*OFM/\!/\mathrm{SO}(\mathbb{R}^d)$ of groupoids over LM defined by $F \mapsto F(0)$ on objects and by $(F, f) \mapsto (F(0), f(0))$ on morphisms, both over $\phi = \pi \circ F \in LM$. This can then be composed with the morphism $\mathrm{ev}^*OFM/\!/\mathrm{SO}(\mathbb{R}^d) \to LM_{\mathrm{dis}}$ (recall that LM_{dis} is the groupoid $LM \rightrightarrows LM$) to produce a morphism $P \to LM_{\mathrm{dis}}$.

It is easily seen that $P_1 \rightrightarrows P_0$ is a Fréchet Lie groupoid over LM; the charts $\kappa_{F_0}\colon L(B_\varepsilon(0)) \to U \subseteq LM$ lift to maps $\kappa_{F_0 *}\colon L(B_\varepsilon(0)) \to P_0$, which defines a local trivialisation $P|_U \simeq U \times \Gamma$. In particular, $\pi\colon P \to LM_{\text{dis}}$ is a submersion on objects.

The groupoid map $P \to LM_{\text{dis}}$, however, does *not* appear to be a principal Γ-2-bundle in the sense of [NW13]. One would like to define a right action $P \times \Gamma \to P$ of Γ on P by $(F \cdot p)(\theta) = F(\theta) \circ p(\theta)$ on objects and by $(F, f) \cdot (p, g) = (F \cdot p, \alpha_p^{-1}(f) \cdot g)$ on morphisms, but $\alpha_p^{-1}(f)$ is not necessarily closed if $f(0) \neq \mathbf{1}$.

In order to see a meaningful 2-group action, we therefore factor the morphism $P \to LM_{\text{dis}}$ through morphisms $P \to \text{ev}^*OFM/\!/\text{SO}(\mathbb{R}^d)$ and $\text{ev}^*OFM/\!/\text{SO}(\mathbb{R}^d) \to LM_{\text{dis}}$. The former is defined on P_0 by $F \mapsto (\pi \circ F, F(0))$, and on P_1 by $(F, f) \mapsto (\pi \circ F, F(0), f(0))$. We set P^{res} to be the preimage of $\text{ev}^*OFM_{\text{dis}} < \text{ev}^*OFM/\!/\text{SO}(\mathbb{R}^d)$, i.e. the subgroupoid of P where the morphisms are restricted to have $f(0) = \mathbf{1}$, i.e., $P_0^{\text{res}} = P_0$, $P_1^{\text{res}} = \Omega\text{Spin}$.

One checks that

$$\begin{array}{ccc} P^{\text{res}} \times K \times K & \xrightarrow{\rho \times \text{Id}} & P^{\text{res}} \times K \\ {\scriptstyle \text{Id} \times \mu}\downarrow & & \downarrow{\scriptstyle \rho} \\ P^{\text{res}} \times K & \xrightarrow{\rho} & P^{\text{res}} \end{array}$$

is strictly commutative, and that $P^{\text{res}} \times K \to P^{\text{res}} \times_{\text{ev}^*OFM} P^{\text{res}}$ is a weak equivalence (even an isomorphism). Thus the groupoid map $P^{\text{res}} \to \text{ev}^*OFM_{\text{dis}}$ is a smooth, strict principal K-2-bundle over ev^*OFM in the sense of [NW13].

Note that the Levi-Civita connection provides a smooth *global* section s^∇ of the principal $P_e\text{SO}(\mathbb{R}^d)$-bundle $P_0 \to \text{ev}^*OFM$, defined by $s^\nabla(\phi, f_0)(\theta) := \exp_{\theta,0}^{\nabla,\phi} f_0$. This gives rise to the trivialisation

$$\text{ev}^*OFM \times P_e\text{SO}(\mathbb{R}^d) \to P_0\colon (\phi, f_0; g) \mapsto s^\nabla(\phi, f_0)g.$$

In short: given a connection on M, we can write any framing F with $F(0) = f_0$ as a covariantly constant framing F^∇ starting at f_0 times a path $g \in P_e\text{SO}(\mathbb{R}^d)$. Since a morphism in P_1 is determined by its source and target, we thus obtain a trivialisation $\text{ev}^*OFM \times K \simeq P$ of Lie groupoids over ev^*OFM.

6.5 The spinor 2-bundle

We define the 'spinor 2-bundle' as a certain Fréchet Lie groupoid $S_1 \rightrightarrows S_0$ over $\text{ev}_0{}^*OFM/\!/\text{SO}(\mathbb{R}^d)$, and thus over LM_{dis}.

6.5.1 As a groupoid

We will equip S with a smooth structure in due time, but first we will describe it as a groupoid *tout court*.

We describe S_0. As we have seen, a framing $F \in P_0$ of ϕ yields a unitary $\mathbb{R}$-action on $\Gamma_{L^2}(\phi^*TM) = T_\phi^{\text{cl}}LM$ by $(T_t^F s)(\theta) = F_\theta^{-1} F_{\theta+t} s(\theta + t)$. If $\Gamma_{L^2}(\phi^*TM)_\mathbb{C} = V_F^+ \oplus V_F^-$ is the polarisation that comes from the positive and negative spectral projections of the generator $-i\partial_t T_t$, then we define $\mathcal{F}_F :=$

$\overline{\wedge V_F^+}$ with the conformal net M_F, where $M_F(I) \subset B(\mathcal{F}_F)$ is the bicommutant of $\mathrm{Cl}(\Gamma(\phi^*TM|_I))$ w.r.t. the representation $\mathcal{F}_F$. Evidently, the isomorphism $F\colon L^2(S^1, \mathbb{R}^d) \to T^{\mathrm{cl}}LM$ respects the polarisation, i.e. it maps $V_0^\pm$ to $V_F^\pm$. This induces a canonical isomorphism $U_F\colon \mathcal{F}_0 \to \mathcal{F}_F$, (not just up to scalar!) and $M_F(I) := U_F M_0(I) U_F^{-1}$. (It is a morphism of sectors.)

Let $\phi \in LM$, $f_0 \in F_{\phi(0)}M$. Let M_ϕ be the conformal net that arises in the way described above for the frame $F_\theta^\nabla := \exp_{\theta,0}^{\nabla,\theta} f_0$ that is covariantly constant along ϕ for the Levi-Civita connection. Then $M_\phi(I)$ is the bicommutant of the *same* algebra $\mathrm{Cl}(\Gamma(\phi^*TM|_I))$, but with respect to a *different* representation $\mathcal{F}_{\phi,f_0}$. (The net does not depend on the choice of f_0, but the representation does.)

We then consider M_F as a $M_\phi - M_\phi$-defect by giving natural transformations $N_W\colon M_\phi|_{[0,\pi]} \to M_F|_{[0,\pi]}$ and $N_B\colon M_\phi|_{[\pi,2\pi]} \to M_F|_{[\pi,2\pi]}$. Because F and F^∇ are frames over the same loop, continuous outside zero, they differ by the path $\Delta F := F^\nabla F^{-1} \in P\mathrm{SO}(\phi^*TM)$ that is continuous outside 0. If, furthermore, $F(0) = f_0$, then $\Delta F \in P_e\mathrm{SO}(\phi^*TM)$. This yields an automorphism $\alpha_{\Delta F}$ of $\mathrm{Cl}(T_\phi^{\mathrm{cl}} LM_{\mathbb{C}})$ that extends to an isomorphism $M_\phi(I) \to M_F(I)$ on any interval $I \subset S^1$ *the closure of which does not contain 0*. We thus see that $\mathcal{F}_{F,f_0}$ is a sector for the M_ϕ-M_ϕ defect M_F. (We write $\mathcal{F}_F$ for $\mathcal{F}_{F,f_0}$ if $F(0) = f_0$, i.e., if the framing F and the covariantly constant framing F^∇ both start at the same frame $f_0 \in OF_{\phi(0)}M$.)

Now the fibre S_0^{ϕ,f_0} of S_0 over (ϕ, f_0) is the set of M_ϕ-M_ϕ defects obtained in this way,

$$S_0^{\phi,f_0} := \{([M_F], \mathcal{F}_{F,f_0})\,;\, F \in POFM\,,\, \pi \circ F = \phi\}\,.$$

Note that every such sector can be canonically identified with the ‘standard’ sector $\mathcal{F}_g$ for the $M_0 - M_0$ defect class $[D_g]$ with $g = F^{-1}F^\nabla$ if we identify M_F with the standard net M_0 through the canonical isomorphism $U_F\colon \mathcal{F}_0 \to \mathcal{F}_F$. The action of M_ϕ (identified with M_0 through $\mathcal{F}_0 \to \mathcal{F}_{\phi,f_0}$) on $\mathcal{F}_0$ (identified with $\mathcal{F}_F$, not $\mathcal{F}_{\phi,f_0}$) is given by the twist α_g, with $g = F^{-1}F^\nabla \in P\mathrm{SO}(\mathbb{R}^d)$.

We describe S_1. It is defined to be the set of intertwiners between the sectors $\mathcal{F}_{F,f_0}$ and $\mathcal{F}_{F',f_0'}$ of the class of M_ϕ-M_ϕ defects $[M_F] = [M_{F'}]$, where F and F' are framings of the same loop ϕ, possibly starting at different frames. Since $([M_\phi], \mathcal{F}_{F,f_0})$ can be canonically identified with $([D_g], \mathcal{F}_g)$ for $g \in P\mathrm{SO}(\mathbb{R}^d)$ given by $g(\theta) = F(\theta)^{-1} \exp_{\theta,0}^{\nabla,\phi} f_0$ (and likewise for F', f_0'), we see that such intertwiners exists if and only if g and g' have the same discontinuity at zero, i.e., if $F(2\pi)^{-1}\mathrm{hol}(\phi)F(0) = F'(2\pi)^{-1}\mathrm{hol}(\phi)F'(0)$, and if, furthermore, the continuous path $g'g^{-1} \in L\mathrm{SO}(T_{\phi(\theta)}M)$ has even $\pi_1(g'g^{-1}) \in \pi_1(\mathrm{SO}(\mathbb{R}^d))$.

We then have an even *projective* unitary transformation $[U_h]\colon P(\mathcal{F}_F, f_0) \to P(\mathcal{F}_{F',f_0'})$. Any choice $\widehat{U}_h \in [U_h]$ yields a unitary $\mathcal{F}_F \to \mathcal{F}_{F'}$ that intertwines the two sectors, and all intertwiners are of this form. Using the canonical identification with $([D_g], \mathcal{F}_g)$ and $([D_g'], \mathcal{F}_{g'})$, as well as the fact that $\mathcal{F}_g = \mathcal{F}_{g'} = \mathcal{F}_0$, we obtain a 1:1-correspondence between $\mathrm{Hom}(\mathcal{F}_{F,f_0}, \mathcal{F}_{F',f_0'})$ and the preimage of $h = g'g^{-1} \in L\mathrm{SO}(\mathbb{R}^d)_+$ under $\widehat{L\mathrm{Spin}} \to L\mathrm{SO}(\mathbb{R}^d)_+$.

The groupoid $S_1 \rightrightarrows S_0$, then, has $\mathrm{Hom}(\mathcal{F}_{F,f_0}, \mathcal{F}_{F',f_0'}) \neq \emptyset$ if and only if $\pi \circ F = \pi \circ F' = \phi \in LM$, and if moreover $h = g'^{-1} g \in P\mathrm{SO}(\mathbb{R}^d)$ is closed and even. In this case $\mathrm{Hom}(\mathcal{F}_{F,f_0}, \mathcal{F}_{F',f_0'})$ is a $U(1)$-torsor canonically isomorphic with the preimage of h in $\widehat{L\mathrm{Spin}}$.

The map $S_0 \to LM\colon ([M_F], \mathcal{F}_{F,f_0}) \mapsto \pi \circ F$ and the map $S_1 \to \mathrm{ev}^* OF \times \mathrm{SO}(\mathbb{R}^d)$ defined by $([M_F], \mathcal{F}_{F,f_0}, \widehat{U}_h) \mapsto (F(0), h(0))$ constitute a morphism onto the action groupoid $\mathrm{ev}^* OFM//\mathrm{SO}(\mathbb{R}^d)$, which then of course maps further into LM_{dis}.

6.5.2 As a smooth 2-bundle

We endow the above groupoid with a smooth structure. Recall that $r\colon \mathrm{ev}^* OFM \to LM$ is the pullback of $OFM \to M$ under $\mathrm{ev}_0\colon LM \to M$. Elements $([M_\phi], \mathcal{F}_{F,f_0}) \in S_0$ are in bijective correspondence to elements $(F, f_0) \in r^* P_0$ in such a way that the fiber maps to $\mathrm{ev}^* OFM$ are respected. We now simply define the Fréchet manifold structure of S_0 to be the one coming from $r^* P_0$.

Recall that the Levi-Civita connection provides a smooth global section $s^\nabla(\phi, f_0)(\theta) := \exp^{\nabla,\phi}_{\theta,0} f_0$ of the principal $P_e\mathrm{SO}(\mathbb{R}^d)$-bundle $P_0 \to \mathrm{ev}^* OFM$. This gives rise to the trivialisation $\mathrm{ev}^* OFM \times P_e\mathrm{SO}(\mathbb{R}^d) \to P_0\colon (\phi, f_0; p) \mapsto s^\nabla(\phi, f_0)p$. In short: given a connection on M, we can write any framing F as a covariantly constant framing F^∇ times a path $g \in P_e\mathrm{SO}(\mathbb{R}^d)$. We likewise trivialise $r^* P_0 \simeq \mathrm{ev}^* OFM \times P\mathrm{SO}(\mathbb{R}^d)$ by $(F, f_0) \mapsto (f_0, p)$ iff $F = \exp^{\nabla,\phi}_{\theta,0} f_0 p$.

If $F = F^\nabla p$, then the sector $\mathcal{F}_{F,f_0}$ for the M_ϕ-M_ϕ defect class $[M_F]$ can be canonically (through the polarisation-preserving isomorphism $L^2(S^1, \mathbb{R}^d) \to T^{\mathrm{cl}}_\phi LM$) identified with the sector $\mathcal{F}_p$ for the M_0-M_0 defect class $[D_p]$. We use the resulting bijective correspondence between S_1 and $r^* P_0 \times \widehat{\Omega\mathrm{Spin}}$ to define a smooth structure on the former.

Remark 14. This, of course, is cheating. What we really want is that the conformal nets M_ϕ, the defect class $[M_F]$ and the sector $(\mathcal{F}_{F,f_0})$ vary 'smoothly' in their parametrisation (ϕ, f_0, g). To define what this means, we probably have to introduce a stack structure on the (weak!) 2-group G of defects, sectors and intertwiners. One would then construct S as the associated bundle to P along the obvious 'smooth' homomorphism $\Gamma \to G$, the smoothness somehow coming from the 'smoothness' of $\rho\colon \Omega\mathrm{Spin} \to \mathrm{Aut}(M_0(S^1_L))$. Since the latter does appear to possess any obvious manifold structure, 'smoothness' should probably be expressed by requiring that the map $\Omega\mathrm{Spin} \times M_0(S^1_L) \times \mathcal{F}_0 \to \mathcal{F}_0$ defined by $(g, X, \psi) \mapsto \rho(g)(X)\psi$ be smooth in g and ψ as well as continuous in A. (Or a stronger requirement for a subalgebra of $M_0(S^1_L)$.) At the end of the day, this should all work because the representation of $\widehat{\Omega\mathrm{Spin}}$ on $\mathcal{F}_0$ has a dense space of smooth vectors.

Using the connection, we thus have $S_0 \simeq \mathrm{ev}^* OFM \times P\mathrm{SO}(\mathbb{R}^d)$ and $S_1 \simeq \mathrm{ev}^* OFM \times P\mathrm{SO}(\mathbb{R}^d) \times \widehat{\Omega\mathrm{Spin}}$, or, more succinctly, $S = \mathrm{ev}^* OFM_{\mathrm{dis}} \times \Gamma$. Recall that coordinate patches $\kappa_\alpha\colon U_\alpha \to LM$ on LM are determined by a choice of framing $F_\alpha \in LOFM$ over $\phi_\alpha \in LM$. The trivialisation induces a local section $f_\alpha\colon U_\alpha \to r^*P_0$ by sending ϕ to (F, f_0) with F the pushforward of F_α along the geodesic flow, and $f_0 = F(0)$. Using the above trivialisation, we obtain smooth maps $f_\alpha\colon U_\alpha \to K_0 < \Gamma_0$. We now *choose* smooth maps $g_{\alpha\beta}\colon U_{\alpha\beta} \to \hat{K}_1$ such that $s \circ g_{\alpha\beta} = f_\beta$ and $t \circ g_{\alpha\beta} = f_\alpha$ (there is something to show here as $U_{\alpha\beta}$ need not be contractible). Let $c_{\alpha\beta\gamma}\colon U_{\alpha\beta\gamma} \to U(1)$ be uniquely determined by $c_{\alpha\beta\gamma}g_{\alpha\gamma} = g_{\beta\gamma}g_{\alpha\beta}$. One checks that $(\delta c)_{\alpha\beta\gamma\delta} := c_{\alpha\beta\gamma}c^{-1}_{\alpha\beta\delta}c_{\alpha\gamma\delta}c^{-1}_{\beta\gamma\delta} = 0$ so that $[c]$ defines a class in $\in \check{H}^2(LM, U(1)) \simeq \check{H}^3(LM, Z)$. Then $[c] = 0$ is necessary and sufficient for the existence of $g_{\alpha\beta}$ satisfying the cocycle identity, and thus to the existence of a class $[\{f_\alpha\}, \{g_{\alpha\beta}\}] \in \check{H}^0(LM, \hat{K})$, i.e. a principal $\hat{K}$-bundle over LM – the string structure. In particular, composing $g_{\alpha\beta}$ with the projection $\hat{K}_1 \to \widehat{\Omega \mathrm{Spin}}$, we obtain the transition functions for the Spin^c cover of $LOFM \to LM$.

6.5.3 The Clifford action

From this string structure, of course, we recover the spinor bundle $\mathcal{F} \to LM$ over LM as

$$\mathcal{F}_\phi := \left\{ \psi \in \bigoplus_{U_\alpha \ni \phi} \mathcal{F}_{F_\alpha} \, ; \, g_{\alpha\alpha'}(\phi)\psi_{\alpha'} = \psi_\alpha \, \forall \alpha, \alpha' \right\} .$$

Because $g_{\alpha\beta}(\phi)$ was chosen to be an intertwiner of defects, the fibre $\mathcal{F}_\phi$ carries naturally the structure af a $[M_\phi]$-$[M_\phi]$ defect. Now although the underlying vector bundle $\mathcal{F} \to LM$ is a purely topological object, the defect classes $[M_\phi]$ encode geometric information. Indeed, the defect class $[M_\phi]$ determines the holonomy of the connection, and from the resulting section of $\mathrm{ev} * \mathrm{SO}(TM) \to LM$, the Levi-Civita connection can be recovered. The spinor bundle $\mathcal{F} \to LM$, then, equipped with the action of the $[M_\phi]$-$[M_\phi]$ defect, appears to be as good an analogue of the finite dimensional spinor bundle with Clifford action as one can expect.

6.5.4 Constant loops

We consider the restriction of $\mathcal{F} \to LM$ to the constant loops $M \subseteq LM$. From a Clifford algebra point of view, t his is not particularly exciting; because the holonomy of a constant loop is zero, the pullback of $\mathcal{F} \to LM$ along the 'constant loop map' $M \to LM$ carries a representation of the trivial defect.

Denote by $\phi_m\colon S^1 \to M$ the constant map with value m. We obtain a neighbourhood of ϕ_m in LM by choosing a constant framing $F_m \in OF_mM$ of the point, and then flowing in the direction of a vector field $F_m \circ v \in \Gamma(\phi_m^* TM) = C^\infty(S^1, T_mM)$, with $v \in C^\infty(S^1, \mathbb{R}^d)$. The intersection of this neighbourhood with M is given by the constant vector fields. The coordinate patches are thus geodesic neighbourhoods of m in M, where each point $m' = \exp_m(v)$ is equipped with the frame $F_{m'}$ that arises by parallel transport along the path $t \mapsto \exp_m(tv)$.

A frame F_m yields an isomorphism $F_m\colon C^\infty(S^1,\mathbb{R}^d)_\mathbb{C} \to C^\infty(S^1, T_mM)_\mathbb{C}$ which maps $V_0^+ = \mathbb{C}^k \oplus (\bigoplus_{n=1}^\infty z^n\mathbb{C}^d)$ to $F_m(\mathbb{C}^k)\oplus(\bigoplus_{n=1}^\infty z^nT_mM)$ (I assume $d = 2k$ even and write z^n for $\theta \mapsto e^{in\theta}$). We write out $\mathcal{F}_0 := \bigwedge V_0^+$ as $\mathcal{F}_0 = \bigwedge \mathbb{C}^k \hat{\otimes} \hat{\bigotimes}_{n=1}^\infty \bigwedge(z^n\mathbb{C}^d)$ (where $\bigwedge z^n\mathbb{C}^d := \bigoplus_{j=0}^d z^{jn}\bigwedge^j \mathbb{C}^d$). If a string structure exists, then we can lift the A change of coordinates, corresponding to a change of constant framing $\tilde{F}_m = F_m g(m)$, yields transition functions $g\colon U_{\alpha\beta} \in \mathrm{SO}(\mathbb{R}^d)$. Although these are canonically implemented on the strictly positive energy part (z^n with $n > 0$), one needs a lift to Spin^c to have $g(m)$ act on the spinor representation $\mathbb{C}^k$. The existence of a string structure guarantees that such a lift exists, and the corresponding bundle over M is (the closure of)

$$\mathcal{F} = \mathbb{S}\,\hat{\otimes}\left(\widehat{\bigotimes}_{n=1}^{\infty} \bigwedge(z^nTM)\right)$$

where z^n can be thought of as the trivial line bundle that caries the $U(1)$-representation $z \mapsto z^n$. Up to a factor $z^{d/24}$, this is formula (24) in [Wit88]. On this bundle of Fock spaces, we have at the point m the defect-free action of the net $M_{\phi_m}(I) = \mathrm{Cl}(C^\infty(I, T_mM))''$.

6.6 Outlook: Fusion structure

Less trivial mathematics is to be expected if we bring into play the multiplicative structure on the 2-group of defects.

Let ϕ and χ be two loops in LM such that $\tau^*\phi_R = \chi_L$, where $\tau\colon [\pi, 2\pi] \to [0, \pi]$ is the time-reversing map $\tau(t) = 2\pi - t$. Let $\phi * \chi$ be the loop (ϕ_L, χ_R).

If ϕ and χ lie in the same co-ordinate patch, then the framing over ϕ_R agrees with the one over χ_L. This means that under the trivialisation, $([M_\phi], \mathcal{F}_\phi)$ corresponds to $([D_g], \mathcal{F}_g)$ and $([M_\chi], \mathcal{F}_\chi)$ to $([D_h], \mathcal{F}_h)$, where $h_L = \tau^* g_R$. It follows from Proposition 19 that the fusion product $\mathcal{F}_g \boxtimes \mathcal{F}_h$ over the algebra $M_\phi([\pi, 2\pi]) \simeq M_\chi([0, \pi])$ is isomorphic to $\mathcal{F}_{\overline{g}_L^{-1}h_R}$, which by left multiplication with the closed loop $g_L\overline{g}_L$ is isomorphic to $\mathcal{F}_{\overline{g}_Lh_R}$. The sector $([D_{\overline{g}_Lh_R}], \mathcal{F}_{\overline{g}_Lh_R})$, of course, is precisely the one corresponding to $\mathcal{F}_{\phi*\chi}$.

If ϕ and χ do not lie in the same patch, then an isomorphism $\mathcal{F}_\phi \boxtimes \mathcal{F}_\chi \to \mathcal{F}_{\phi*\chi}$ still exists. This is because even though g_R and h_L no longer agree, we still have $\mathrm{hol}(\phi)\mathrm{hol}(\psi) = \mathrm{hol}(\phi * \psi)$, which is all that matters.

Now suppose that we have 4 paths $q_i\colon [0, \pi] \to M$ with the same start and endpoint, and we set $\phi_{ij} = (p_i, \tau^{-1*}p_j)$. We choose isomorphisms $\alpha_{ijk}\colon \mathcal{F}_{\phi_{ij}} \boxtimes \mathcal{F}_{\phi_{jk}} \to \mathcal{F}_{ik}$ (which are unique only up to $U(1)$), and let $c_{p_1,p_2,p_3,p_4} \in U(1)$ be the difference between $\alpha_{134} \circ (\alpha_{123} \otimes \mathrm{Id})$ with $\alpha_{124} \circ (\mathrm{Id} \otimes \alpha_{234}) \circ a_{1234}$, where

$$a_{1234}\colon (\mathcal{F}_{12} \boxtimes \mathcal{F}_{23}) \boxtimes \mathcal{F}_{34} \to \mathcal{F}_{12} \boxtimes (\mathcal{F}_{23} \boxtimes \mathcal{F}_{34})$$

is the appropriate associator for the CFP, cf. Thm 26. If c can be made to vanish for an appropriate choice of α, we will say that the string structure $\mathcal{F}$ admits a fusion product.

Theorem 26 appears to imply that within each coordinate patch, it is possible to choose α so as to trivialise c. In the presence of such a 'Poincaré Lemma', then, the vanishing of c becomes a purely toplogical problem.

The next thing to do is to formulate a cohomology group where $[c]$ lives, and a way to obtain the class $[c]$ from the topology of M alone. It seems worth while to reformulate Thm. 26 and Prop. 19 in terms of loops living on the manifold; this should simplify matters in that one can identify explicitly, by its generators on $\mathcal{F}_\phi$ and $\mathcal{F}_\chi$ alike, the algebra of operators over which one takes the fusion product. It may well be that the obstruction vanishes automatically.

Also, I should probably take a look at condition (iv) on page 4 of [Wal10], and see how it ties in with the modular operators, Cor. 10.

7 Fusion structure

Let (M, g) be an orientable riemannian manifold of dimension n, and denote by $F \to M$ the principal $\mathrm{SO}(n)$-bundle of orthogonal frames. We will construct a bundle of defects over LM that generalises the Clifford bundle $\mathrm{Cl}(TM) \to M$ of a finite dimensional manifold.

If, moreover, both M and LM are spin, then we interpret the spinor bundle over LM as a bundle of sectors.

Finally, if M is string, we define a fusion structure on the spinor bundle.

7.1 Clifford bundles over LM

We first define the loop space analogue of what for finite dimensional Riemannian manifolds is the bundle $\mathrm{Cl}(TM_{\mathbb{C}}, g_{\mathbb{C}}) \to M$ of Clifford algebras.

We start by defining the *Clifford C^*-algebra* $\mathrm{Cl}(\mathcal{H}_{\mathbb{R}}, G)$ canonically associated to a *real* Hilbert space $(\mathcal{H}_{\mathbb{R}}, G)$. The real Clifford algebra $\mathrm{Cl}_0(\mathcal{H}_{\mathbb{R}}, G)$ is defined as $T(\mathcal{H}_{\mathbb{R}})/I(G)$ with $I(G)$ the ideal generated by the Clifford relations $\{h \otimes h' + h' \otimes h - G(h, h')\}$. Its complexification is made into a $*$-algebra by the antilinear involution $\psi_1 \cdot \ldots \cdot \psi_n \mapsto \overline{\psi_n} \cdot \ldots \cdot \overline{\psi_1}$. It has a canonical faithful $*$-representation (the left regular representation) on the Hilbert space generated by the exterior algebra $\bigwedge \mathcal{H}_{\mathbb{C}}$, where the scalar product is induced by the sesquilinear extension of G to $\mathcal{H}_{\mathbb{C}}$. The action is given by $c(\psi) = \frac{1}{\sqrt{2}}(a(\psi) + a^*(\psi))$, where the creation and annihilation operators act as usual (cf. [BR81, §5.2]),

$$\begin{aligned} a^*(\psi)\psi_1 \wedge \ldots \wedge \psi_n &= (n+1)^{1/2} \psi \wedge \psi_1 \wedge \ldots \wedge \psi_n \,, \\ a(\psi)\psi_1 \wedge \ldots \wedge \psi_n &= n^{1/2} \sum_{i=1}^{n} (-1)^{i-1} (\psi, \psi_i) \psi_1 \wedge \ldots \wedge \hat{\psi}_i \wedge \ldots \wedge \psi_n \,. \end{aligned}$$

We now define the Clifford C^*-algebra $\mathrm{Cl}(\mathcal{H}_{\mathbb{R}}, G)$ to be the norm closure of $\mathrm{Cl}_0(\mathcal{H}_{\mathbb{R}}, G) \otimes_{\mathbb{R}} \mathbb{C}$ in $B(\overline{\bigwedge \mathcal{H}_{\mathbb{C}}})$ (or, indeed, in any other representation if $\dim(\mathcal{H}_{\mathbb{R}})$ is even or infinite [SS64]). It has the universal property that every $\mathbb{R}$-linear map $c \colon \mathcal{H}_{\mathbb{R}} \to \mathcal{A}$ into a unital C^*-algebra $\mathcal{A}$ that satisfies $c(\psi)^* = c(\psi)$ and $\{c(\psi), c(\chi)\} = G(\psi, \chi)\mathbf{1}$ factors through a unique homomorphism $\mathrm{Cl}(\mathcal{H}_{\mathbb{R}}, G) \to$

$\mathcal{A}$ of C^*-algebras. In particular, we have a functor Cl from the category of Hilbert spaces with isometries to the category of C^*-algebras with $*$-homomorphisms. One checks that the $O(\mathcal{H}_{\mathbb{R}})$-action

$$O(\mathcal{H}_{\mathbb{R}}) \times \mathrm{Cl}(\mathcal{H}_{\mathbb{R}}, G) \to \mathrm{Cl}(\mathcal{H}_{\mathbb{R}}, G)$$

coming from the group homomorphism $\mathrm{Cl}\colon O(\mathcal{H}_{\mathbb{R}}) \to \mathrm{Aut}(\mathrm{Cl}(\mathcal{H}_{\mathbb{R}}, G))$ is continuous if both $O(\mathcal{H}_{\mathbb{R}})$ and $\mathrm{Cl}(\mathcal{H}_{\mathbb{R}}, G)$ are equipped with the norm topology.

We now construct such a Clifford C^*-algebra for each point in loop space, and show that the resulting algebras fit together to a *continuous* bundle of C^*-algebras.

The tangent space $T_\phi LM = \Gamma(\phi^*TM)$ at the loop $\phi \in LM$ carries an inner product

$$G_\phi(s, s') := \frac{1}{2\pi} \int_0^{2\pi} g_{\phi(\theta)}(s(\theta), s'(\theta)) d\theta,$$

which is smooth and nondegenerate. It is invariant[4] under the $\mathbb{R}$-action on $T_\phi LM$ defined by parallel transport along the Levi-Civita connection, $(T_t s)(\theta) := \exp_t^{\nabla,\phi}(s(\theta - t))$ where $\exp_t^{\nabla,\phi}\colon T_{\phi(\theta-t)}M \to T_{\phi(\theta)}M$ is parallel transport along the path $\phi|_{[\theta-t,t]}$.

For each subinterval $I \subset S^1$, we define the real Hilbert space $L^2(I, \phi^*TM)$ to be the closure of $\Gamma(\phi^*TM|_I)$ w.r.t. G_ϕ, and we define $\mathrm{Cl}_\phi(I)$ to be the Clifford C^*-algebra $\mathrm{Cl}(L^2(I, \phi^*TM), G)$. We set $\mathrm{Cl}(I) := \bigsqcup_{\phi \in LM} \mathrm{Cl}_\phi(I)$, and we give the projection $\mathrm{Cl}(I) \to LM$ the structure of a continuous bundle of C^*-algebras over LM as follows.

Since the homomorphism $L\mathrm{SO}(n) \to O(L^2(I, \mathbb{R}^n))$ induced by the action of $\mathrm{SO}(n)$ on $\mathbb{R}^n$ is norm continuous, so is the action of $L\mathrm{SO}(n)$ on $\mathrm{Cl}(L^2(I, \mathbb{R}^n))$. The bundle $\mathcal{A}(I) := LFM \times_{L\mathrm{SO}(n)} \mathrm{Cl}(L^2(I, \mathbb{R}^n))$ is thus a continuous bundle over LM such that every fibre $\mathcal{A}_\phi(I)$ over $\phi \in LM$ is canonically a C^*-algebra. We now identify $\mathcal{A}_\phi(I)$ with $\mathrm{Cl}_\phi(I)$. For every framing $f \in LFM$ (with $FM \to M$ the bundle of orthogonal frames), the induced isometry $f\colon L^2(I, \mathbb{R}^n) \to L^2(I, \phi^*TM)$ yields an isomorphism $\mathrm{Cl}(f)\colon \mathrm{Cl}(L^2(I, \mathbb{R}^n)) \to \mathrm{Cl}_\phi(I)$ of C^*-algebras. We now define the isomorphism $\mathcal{A}_\phi(I) \to \mathrm{Cl}_\phi(I)$ by $[f, a] \mapsto \mathrm{Cl}(f)(a)$. This is well defined because if two framings differ by a loop $g \in L\mathrm{SO}(n)$, that is, if $f'(\theta) = f(\theta) \circ g(\theta)$, then $\mathrm{Cl}(f \circ g) = \mathrm{Cl}(f)\mathrm{Cl}(g)$, so that $[f \circ g, \mathrm{Cl}(g^{-1})(a)]$ maps to $\mathrm{Cl}(f)(a)$ as it should.

The C^*-algebra $\Gamma_{\mathrm{ct}}\mathcal{A}(I)$ of continuous sections together with the homomorphisms $\pi_\phi\colon \Gamma_{\mathrm{ct}}\mathcal{A}(I) \to \mathrm{Cl}_\phi(I)$ defined by evaluation at ϕ composed with the isomorphism $\mathcal{A}_\phi(I) \to \mathrm{Cl}_\phi(I)$ then makes $\mathrm{Cl}(I) \to LM$ into a continuous bundle of C^*-algebras.

Now note that although the C^*-algebras $\mathrm{Cl}_\phi(I)$ do not come with a canonical representation, they carry canonical strong, weak and ultraweak topologies

[4] It is tempting to write '$d\phi(\theta)$' rather than '$d\theta$' in the integrand, in which case the completion is effectively the space of square integrable vector fields on $\mathrm{Im}(\phi)$. This would be invariant under the pushforward of the S^1-action on LM by $R_t\phi(\theta) := \phi(\theta - t)$ which, of course, is not fibre preserving. Since we wish to encode the holonomy of the metric, we have chosen not to go down this path.

and can thus be completed to W^*-algebras. Indeed, since the unitary $\mathbb{R}$-action by parallel transport on the complexification $\mathcal{H}_{\mathbb{C}}$ of $\mathcal{H}_{\mathbb{R}} := L^2(S^1, \phi^*TM)$ commutes with complex conjugation, the orthogonal decomposition $\mathcal{H}_{\mathbb{C}} = V_+ \oplus V_0 \oplus V_-$ into positive, zero and negative part of the spectrum of its generator $-i\nabla_{\phi_*\partial_\theta}$ satisfies $\overline{V_+} = V_-$, $\overline{V_-} = V_+$ and $\overline{V_0} = V_0$. In particular, $V_+ \oplus V_-$ is the complexification of $\mathcal{H}_{\mathbb{R}}/(\mathcal{H}_{\mathbb{R}} \cap V_0)$, and V_0 of $V_0 \cap \mathcal{H}_{\mathbb{R}} := V_{0,\mathbb{R}}$. Although we do not have a canonical polarisation for $V_0 \neq \{0\}$, the space V_0 of covariantly constant sections of $\phi^*TM \to S^1$ is always finite dimensional, so that the *class* of polarisations that are compatible with complex conjugation and Hilbert-Schmidt w.r.t. $V_+ \oplus V_-$ is well defined.

We thus equip $\mathrm{Cl}(\mathcal{H}_{\mathbb{R}}/V_{0,\mathbb{R}})$ with the ultraweak topology w.r.t. its representation on the Fock space $\overline{\bigwedge V_+}$, where the action of $\mathrm{Cl}(\mathcal{H}_{\mathbb{R}}/V_{0,\mathbb{R}})$ is defined by the identification of the Fock space with the Hilbert closure of the quotient

$$\bigwedge V_+ = \mathrm{Cl}_0(\mathcal{H}_{\mathbb{R}}/V_{0,\mathbb{R}})_{\mathbb{C}}/\mathrm{Cl}_0(\mathcal{H}_{\mathbb{R}}/V_{0,\mathbb{R}})_{\mathbb{C}} \cdot V_-$$

of $\mathrm{Cl}_0(\mathcal{H}_{\mathbb{R}}/V_{0,\mathbb{R}})$-modules. This equips the C^*-algebra $\mathrm{Cl}(\mathcal{H}) = \mathrm{Cl}(V_{0,\mathbb{R}})\widehat{\otimes}\mathrm{Cl}(\mathcal{H}_{\mathbb{R}}/V_{0,\mathbb{R}})$ with a topology that depends only on the holonomy of the Levi-Civita connection around ϕ. If we choose any finite dimensional faithful representation $\mathbb{S}$ of $\mathrm{Cl}(V_{0,\mathbb{R}})$, then the ultraweak topology on $\mathrm{Cl}(\mathcal{H}_{\mathbb{R}})$ is the one resulting from its representation on $\mathbb{S} \,\widehat{\otimes}\, \overline{\bigwedge V_+}$.

We pull back this topology to $\mathcal{A}_\phi(I)$. Considering $f \in LFM$ as a trivialisation $f\colon S^1 \times \mathbb{R}^n \to \phi^*TM$, the Levi-Civita connection $\Gamma(\phi_*\partial_\theta) \in \mathfrak{so}(T_{\phi(\theta)}M)$ on ϕ^*TM pulls back to $\partial_\theta + A^f(\theta)$, where the loop

$$A^f(\theta) = f^{-1}f' + f^{-1}(\theta)\Gamma(\phi_*\partial_\theta)f(\theta)$$

in $L\mathfrak{so}(n)$ transforms under $g \in L\mathrm{SO}(n)$ as $A^{fg^{-1}} = gA^f g^{-1} - g'g^{-1}$. The isometry $f\colon L^2(S^1, \mathbb{C}^n) \to L^2(S^1, \phi^*TM_{\mathbb{C}})$ induced by the framing f thus pulls back V_+, V_0 and V_- to V_+^A, V_0^A and V_-^A, the images of the positive, zero and negative spectral projections of the differential operator $\partial_t + A^f$. The pullback of the ultraweak topology from $\mathrm{Cl}_\phi(I)$ to $\mathcal{A}_\phi(I)$ is thus the one coming from the representation $\overline{\bigwedge V_+^A}$ of $\mathrm{Cl}(L^2(S^1, \mathbb{R}^n)/V_{0,\mathbb{R}}^A)$ and the identification $\mathrm{Cl}(L^2(S^1, \mathbb{R}^n)) = \mathrm{Cl}(V_{0,\mathbb{R}}^A)\widehat{\otimes}\mathrm{Cl}(L^2(S^1, \mathbb{R}^n)/V_{0,\mathbb{R}}^A)$ with the tensor product by a finite dimensional algebra. Because $V_+^{A^{fg^{-1}}} = g(V_+^{A^f})$ by gauge invariance, this topology of $\mathcal{A}_\phi(I) = LFM \times_{L\mathrm{SO}(n)} \mathrm{Cl}(L^2(I, \mathbb{R}^n))$ does not depend on the choice of framing $f \in LFM$ with which to identify $\mathcal{A}_\phi(I)$ with $\mathrm{Cl}(L^2(I, \mathbb{R}^n))$, so that $\mathcal{A}_\phi(I)$ caries a well defined ultraweak topology. We denote its closure by $\mathcal{M}_\phi(I)$, and note that for each I, the map $\bigsqcup_{\phi \in LM} \mathcal{M}_\phi(I) \to LM$ is a bundle of W^*-algebras in the sense that every fibre is a W^*-algebra, and every $\mathcal{M}_\phi(I)$ has a C^*-subalgebra $\mathcal{A}_\phi(I)$ such that the $\mathcal{A}_\phi(I)$ constitute a *continuous* bundle.

By construction, the $\mathcal{A}(I)$ include continuously in $\mathcal{A}(S^1)$, so that $I \mapsto \mathcal{M}(I)$ is a functor from the category of subintervals of S^1 to the category of *bundles* of W^*-algebras with the obvious morphisms. We think of this as a *bundle of conformal nets.*

Remark 15. The topology with which we equipped $\mathrm{Cl}_\phi(S^1)$ is *not* the same as the topology from the left regular representation $\overline{\bigwedge \mathcal{H}_\mathbb{C}}$, with $\mathcal{H}_\mathbb{C} := L^2(S^1, \phi^*TM)_\mathbb{C}$, that we used in order to define the C^*-norm. Indeed, it is easy to check (cf. [BR81, § 5.2]) that for any subspace $W \subseteq \mathcal{H}_\mathbb{C}$ of dimension $2k$, the state defined by the vector $\Omega \in \overline{\bigwedge \mathcal{H}_\mathbb{C}}$ on the subalgebra $\mathrm{Cl}(W)\hat{\otimes}\mathbf{1} \subseteq \mathrm{Cl}(\mathcal{H}_\mathbb{C})$ is precisely the normalised trace under the isomorphism $\mathrm{Cl}(W) \simeq \widehat{\bigotimes}_{i=1}^{k} M^2(\mathbb{C})$. This and the isomorphism $\widehat{\bigotimes}_{i=1}^{\infty} M^2(\mathbb{C}) \simeq \mathrm{Cl}(\mathcal{H}_\mathbb{C})$ obtained from an orthonormal basis $\{p_i, q_i\}_{i\in\mathbb{N}}$ of $\mathcal{H}_\mathbb{C}$ show that the von Neumann algebra closure of $\mathrm{Cl}_\phi(S^1)$ (and indeed of any $\mathrm{Cl}_\phi(I)$) w.r.t. the left regular representation is the hyperfinite type II_1 factor. For the representation $\mathbb{S}\,\widehat{\otimes}\overline{\bigwedge V_+}$, the situation is radically different; the closure $\mathrm{Cl}_\phi(S^1)''$ is the full algebbra of bounded operators, a type I_∞ factor, whereas for proper subintervals $I \subset S^1$, $\mathrm{Cl}_\phi(I)''$ is a factor of type III_1.

Remark 16. If the space of loops is equipped with a measure, then the space of L^2 sections of the spinor bundle is the Hilbert space which, presumably, is the natural home for a Dirac operator on loop space. The natural measure on loop space that is canonically associated to the Riemannian metric, however, lives on a space of continuous paths that are not differentiable, but satisfy $|\phi(t+\epsilon) - \phi(t)| \sim \sqrt{\epsilon}$, cf. [AD99]. In order to make sense of our construction in this context, one would have to make sense of parallel transport for paths that are continuous but not C^1. This can presumably be done [Hs88], and one should expect that solutions of the stochastic differential equation $p_t^{-1}dp_t = -\Gamma(\phi_t)d\phi_t$ satisfy $|p_{t+\epsilon} - p_t| \sim 1/\sqrt{\epsilon}$, so that our main tool, proposition 12, is (on the verge of) working. This should connect with the probabilistic proof [Bi84a, Bi84b] of the Atiyah-Singer Index Theorem. All of this is rather beyond the scope of what we are trying to reach in this paper, though.

7.2 The spinor bundle over LM

The nets of bundles of Clifford algebras $\mathcal{A}(I)$ and $\mathcal{M}(I)$ were defined without any assumptions on the topology of M. We would now like to define a spinor bundle over LM in such a way that the fibres carry appropriate representations of the $\mathcal{M}_\phi(I)$.

As in the finite dimensional case, there are topological obstructions to the existence of such bundles. First of all, we require that M be spin, so that the principal $L\mathrm{SO}(n)$ bundle $LF \to LM$ of framings admits a subbundle $L^+F \to LM$ with structure group $L\mathrm{Spin}(n) < L\mathrm{SO}(n)$, namely the loops in F for which the holonomy of the spin cover $Q \to F$ vanishes. We will call these the *even* framings.

By definition, a *spin-structure* for LM is a principal $\widehat{L\mathrm{Spin}}(n)$-bundle $\widehat{LF} \to LM$ with an $\widehat{L\mathrm{Spin}}(n)$-equivariant bundle map $\widehat{LF} \to L^+F$. In other words, a bundle such that the diagram

$$\begin{array}{ccc} \widehat{LF} \times \widehat{L\mathrm{Spin}}(n) & \longrightarrow & L^+F \times L\mathrm{Spin}(n) \\ \downarrow & & \downarrow \\ \widehat{LF} & \longrightarrow & L^+F \end{array}$$

is commutative, where the vertical maps are the group actions and the horizontal maps are the bundle map and its product with the the central extension $\widehat{L\mathrm{Spin}}(n) \to L\mathrm{Spin}(n)$. (Since the kernel of the latter is $U(1)$, the name 'Spin^c-structure' would perhaps have been more appropriate.)

If one chooses lifts $\widehat{g}_{\alpha\beta}\colon U_\alpha \cap U_\beta \to L\mathrm{Spin}(n)$ of the transition functions $g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to L\mathrm{Spin}(n)$, then the cocycle $c_{\alpha\beta\gamma}\colon U_\alpha \cap U_\beta \cap U_\gamma$ defined by $c_{\alpha\beta\gamma} = \widehat{g}_{\alpha\beta}\widehat{g}_{\beta\gamma}\widehat{g}_{\alpha\gamma}^{-1}$ is closed, and exact if and only if a spin structure exists. The obstruction for existence of spin structures is thus the class $[c] \in \check{H}^2(LM, U(1)) \simeq \check{H}^3(LM, \mathbb{Z})$.

Assuming that both M and LM are spin, so that a spin structure $\widehat{LF} \to LM$ exists, we would like to consider the associated spinor bundle $\mathcal{F} \to LM$, defined by $\mathcal{F} := \widehat{LF} \times_{\widehat{L\mathrm{Spin}(n)}} \mathcal{F}_0$, as a bundle of sectors for the net $I \mapsto \mathcal{M}_\phi(I)$ of W^*-algebras.

7.3 Representations of $\mathrm{Cl}(L^2(S^1, \mathbb{R}^n))$ over the connections

Upon a choice of framing $f \in LFM$, we have an identification $\mathcal{A}_\phi(I) \to \mathrm{Cl}(L^2(I, \mathbb{R}^n))$ and a continuous representation of $\mathrm{Cl}(L^2(I, \mathbb{R}^n))$ on the Fock space $\mathcal{F}_A := \mathbb{S} \mathbin{\widehat{\otimes}} \overline{V_+^A}$, with $V_+^A \subset L^2(S^1, \mathbb{C}^n)$ the positive energy space of the antihermitean covariant derivative operator $\nabla^A := \partial_\theta + A$. We now compare these representations for different values of $A \in \Omega^1(S^1, \mathfrak{so}(n))$.

For every $A \in \Omega^1(S^1, \mathfrak{so}(n))$, we define the parallel transport $p \in P_e\mathrm{SO}(n)$, with

$$P_e\mathrm{SO}(n) := \{p \in C^\infty(\mathbb{R}, \mathrm{SO}(n))\,;\, p(0) = \mathbf{1}\,,\, p(\theta + 2\pi) = p(\theta)\ \forall\, \theta \in \mathbb{R}\}\,,$$

by the differential equation $\nabla^A p = 0$, or, equivalently, $(\partial_\theta p)p^{-1} = -A$, with initial condition $p(0) = \mathbf{1}$. This defines a bijection $\Omega^1(S^1, \mathfrak{so}(n)) \to P_e\mathrm{SO}(n)$ which is $L\mathrm{SO}(n)$-equivariant under the action $g\colon A \mapsto gAg^{-1} - g'g^{-1}$ on the former and and $g\colon p(\theta) \mapsto g(\theta)p(\theta)g(0)^{-1}$ on the latter space.

We denote by $h(A) := p(2\pi) \in \mathrm{SO}(n)$ the holonomy of ∇^A. If A is the pullback by $\phi \in LM$ of the Levi-Civita connection on M, then of course $h(A)$ is the holonomy around ϕ of the Levi-Civita connection $\mathrm{Hol}_\phi(\nabla^{LC}) \in \mathrm{SO}(T_{\phi(0)}M)$, pulled back by the frame $f(0)\colon \mathbb{R}^n \to T_{\phi(0)}M$. We denote by $\widehat{h}(A) \in \mathrm{Spin}(n)$ the equivalence class of p in the universal cover $\mathrm{Spin}(n) \to \mathrm{SO}(n)$. If M is spin, it corresponds to the holonomy of the spin lift of ∇^{LC}.

Proposition 27 ([PS86], § 4.3)**.** *The gauge orbits on $\Omega^1(S^1, \mathfrak{so}(n))$ correspond to conjugacy classes of the holonomy, and the stabiliser of a connection corresponds to to the centraliser of its holonomy.*

More precisely, two connections A and A' are $L\mathrm{SO}(n)$-equivalent if and only if $h(A)$ and $h(A')$ are conjugate in $\mathrm{SO}(n)$. *They are $L\mathrm{Spin}(n)$-equivalent if and only if $\widehat{h}(A)$ and $\widehat{h}(A')$ are conjugate in* $\mathrm{Spin}(n)$. *And $g \in L\mathrm{SO}(n)$ stabilises A if and only if $g_0 \in \mathcal{Z}(h(A))$ and $g(\theta) = p(\theta)g(0)p(\theta)^{-1}$.*

Proof. If $g(\theta)p(\theta)g(0)^{-1} = p'(\theta)$ for some $g \in L\mathrm{SO}(n)$, then certainly we have $g(0)h(A)g(0)^{-1} = h(A')$ and the holonomies are conjugate in $\mathrm{SO}(n)$. If $g \in L\mathrm{Spin}(n) < L\mathrm{SO}(n)$ (that is, it has even winding number), then it can be contracted to $g(0)$, so that $[p'] = [g(\theta)p(\theta)g(0)^{-1}] = [g(0)][p][g(0)]^{-1}$ is a conjugacy in $\mathrm{Spin}(n)$.

Now suppose that $h(A') = rh(A)r^{-1}$ for $r \in \mathrm{SO}(n)$. Then the path $g(\theta) := p'(\theta)rp(\theta)^{-1}$ is in fact a loop, so that p and p' are in the same $L\mathrm{SO}(n)$-orbit. It is a loop in $\mathrm{Spin}(n)$ if and only if p and p' have the same winding number.

An element $g \in L\mathrm{SO}(n)$ stabilises a connection A if $g(\theta) = p(\theta)g(0)p(\theta)^{-1}$, so $g \in \mathrm{Stab}(A)$ is uniquely determined by g_0. It is clear that $g_0 = p(2\pi)g(0)p(2\pi)^{-1}$. □

If two connections are in the same $\mathrm{SO}(n)$-orbit, $A' = gAg^{-1} - g'g^{-1}$, then $g(V_+^A) = V_+^{A'}$, so that the isometry $g\colon L^2(S^1, \mathbb{R}^n) \to L^2(S^1, \mathbb{R}^n)$ yields a *canonical* isomorphism $S_g\colon \mathcal{F}_A \to \mathcal{F}_{A'}$ satisfying $\mathrm{Cl}^{A'}(gv) = S_g \mathrm{Cl}^A(v) S_g^{-1}$. Now the $\mathrm{Cl}(L^2(S^1, \mathbb{R}^n))$-representation on $\mathcal{F}_A$ is equivariant for the $\widehat{L\mathrm{SO}(n)}$-representation $\widehat{g} \mapsto U_{\widehat{g}}$ constructed from it, $\mathrm{Cl}^A(gv) = U_{\widehat{g}} \mathrm{Cl}^A(v) U_{\widehat{g}}^{-1}$, so that $S_g U_{\widehat{g}}^{-1}\colon \mathcal{F}_A \to \mathcal{F}_{A'}$ is an intertwiner of Clifford algebra representations. In particular, the topology we have bestowed upon $\mathcal{A}_\phi(I)$ depends only on the conjugacy class of the holonomy. Note that $U_{\widehat{g}}$ is an isomorphism of $\mathbb{Z}/2\mathbb{Z}$-*graded* Hilbert spaces if and only if $g \in L\mathrm{Spin}(n) < L\mathrm{SO}(n)$.

We fix a maximal torus $T \subseteq \mathrm{Spin}(n)$. Because it intersects every conjugacy class nontrivially, conjugacy classes of $\mathrm{Spin}(n)$ (and thus $L\mathrm{Spin}(n)$-orbits in $\Omega^1(S^1, \mathfrak{so}(n))$) correspond to elements of $T/\mathcal{W}$ with $\mathcal{W}$ the Weyl group, or, with $\mathfrak{t} = \mathrm{Lie}(T)$, with elements of the quotient $\mathfrak{t}/\mathcal{W}_{\mathrm{aff}}$ by the affine Weyl group.

If we identify $\Omega^1(S^1, \mathfrak{so}(n)) \oplus \mathbb{R}$ with the smooth part of the dual of $\widehat{L\mathfrak{so}(n)}$ by the pairing $\langle (A, k), (\xi, \lambda) \rangle := k\lambda - \int_{S^1} \kappa(A, \xi)$, where κ is the Killing form normalised so as to induce a generator of $H^3(\mathrm{Spin}(n), \mathbb{Z})$, then the coadjoint action of $L\mathrm{Spin}(n)$ on $\Omega^1(S^1, \mathfrak{so}(n)) \oplus \{1\}$, the affine subspace at level 1, is precisely the one that we used in Proposition (27). From Prop. 4.3.7 and Thm. 9.3.5 of [PS86], it follows that the coadjoint orbits corresponding to irreducible smooth positive energy representations of $T \times \widehat{LG}$ are precisely the ones corresponding to the dominant weights in the affine Weyl chamber intersected with the level 1 hyperplane.

7.4 Bundles of defects over LM

The pullback along $\mathrm{ev}_0\colon LM \to M$ of the oriented orthogonal frame bundle is a smooth principal $\mathrm{SO}(n)$-bundle $\mathrm{ev}_0^* F \to LM$ over LM. Since the homomorphism $\mathrm{SO}(n) \to O(L^2(I, \mathbb{R}^n))$ induced by the action of $\mathrm{SO}(n)$ on $\mathbb{R}^n$ is norm continuous, the bundle $\mathcal{A}(I) := \mathrm{ev}_0^* F \times_{\mathrm{SO}(n)} \mathrm{Cl}(L^2(I,\mathbb{R}^n))$ is a continuous bundle over LM such that every fibre $\mathcal{A}_\phi(I)$ over $\phi \in LM$ is canonically a C^*-algebra.

We identify $\mathcal{A}_\phi(I)$ with $\mathrm{Cl}_\phi(I)$ as follows. We choose a framing $F^\nabla \in L^+F$ that satisfies $F^\nabla(0) = f_0 \in F_{\phi(0)}$ and is covariantly constant w.r.t. the Levi-Civita connection on I. If $1 \notin \overline{I}$, so that $\overline{I} = \exp(i[\theta_0, \theta_1])$ with $\theta_0, \theta_1 \in (0, 2\pi)$, then we require moreover that F^∇ be covariantly constant on the 'left' connected component $\exp(i(0,\theta_0))$ of $S^1 - (\{1\} \cup \overline{I})$. The isometry $F^\nabla \colon L^2(I,\mathbb{R}^n) \to L^2(I, \phi^*TM)$ induced by this framing is thus well defined, and yields an isomorphism $\alpha(f_0, I) \colon \mathrm{Cl}(L^2(I,\mathbb{R}^n)) \to \mathrm{Cl}_\phi(I)$. Since $\alpha(f_0 g, I) = \alpha(f_0, I) \circ \alpha_g$, the expression $[f_0, a] \mapsto \alpha(f_0, I)(a)$ is independent of the representative, and defines an isomorphism $\mathcal{A}_\phi(I) \to \mathrm{Cl}_\phi(I)$. The C^*-algebra $\Gamma_{\mathrm{ct}}\mathcal{A}(I)$ of continuous sections together with the homomorphisms $\pi_\phi \colon \Gamma_{\mathrm{ct}}\mathcal{A} \to \mathrm{Cl}_\phi(I)$ defined by evaluation at ϕ composed with the isomorphism $\mathcal{A}_\phi(I) \to \mathrm{Cl}_\phi(I)$ then makes $\mathrm{Cl}(I) \to LM$ into a continuous bundle of C^*-algebras.

This a *net* of bundles of C^*-algebras in the following sense. Any inclusion $I \subseteq J$ of intervals yields canonical inclusions $\mathrm{Cl}_\phi(I) \hookrightarrow \mathrm{Cl}_\phi(J)$ of C^*-algebras for every $\phi \in LM$. There are 4 possibilities:

1) $1 \in \overline{I}$ and $1 \in \overline{J}$.

2) $1 \notin \overline{I}$ and $1 \notin \overline{J}$.

3) $1 \notin \overline{I}$ and $1 \in \overline{J}$, with $I \subset J_L$ a subset of the 'left' connected component of $\overline{J} - \{1\}$. (That is, $J_L \cap \exp(i[0,\epsilon]) \neq \emptyset$ for all $\epsilon > 0$.)

4) $1 \notin \overline{I}$ and $1 \in \overline{J}$, with $I \subset J_R$ a subset of the 'right' connected component of $\overline{J} - \{1\}$. (The one with $J_R \cap \exp(i[0,-\epsilon]) \neq \emptyset$ for all $\epsilon > 0$.)

In cases 1), 2) and 3), the inclusion $\iota \colon \mathcal{A}_\phi(I) \to \mathcal{A}_\phi(J)$ defined by $[f_0, a] \mapsto [f_0, \iota^0(a)]$ with $\iota^0 \colon \mathrm{Cl}(L^2(I,\mathbb{R}^n)) \to \mathrm{Cl}(L^2(J,\mathbb{R}^n))$ induced by the inclusion $L^2(I,\mathbb{R}^n) \to L^2(J,\mathbb{R}^n)$ of Hilbert spaces is a bundle map $\mathcal{A}(I) \to \mathcal{A}(J)$ with the property that $\pi_\phi \circ \iota \colon \mathrm{Cl}_\phi(I) \to \mathrm{Cl}_\phi(J)$ is precisely the canonical inclusion mentioned above.

In case 4), the framings $F^\nabla(I)$ and $F^\nabla(J)$ do *not* agree on I, but rather differ by the holonomy around $\phi \in LM$ of the Levi-Civita connection. If we set $\mathrm{Hol}(\phi) := \exp^\nabla_{2\pi} \in \mathrm{SO}(T_{\phi(0)}M)$ and $h(\phi, f_0) := f_0^{-1} \exp^\nabla_{2\pi} f_0 \in \mathrm{SO}(n)$, then from $F^\nabla(I) = F^\nabla(J)|_I \cdot h(\phi, f_0)$, it follows that the correct map $\mathcal{A}_\phi(I) \to \mathcal{A}_\phi(J)$ is $\iota \colon [f_0, a] \mapsto [f_0, \alpha_{h(\phi,f_0)} \circ \iota^0(a)]$, the inclusion twisted by the holonomy. This is well defined because $h(\phi, f_0 g) = g^{-1} h(\phi, f_0) g$, and it defines a homomorphism of continuous bundles of C^*-algebras because $\mathrm{Hol} \colon \mathrm{ev}_0{}^* F \to \mathrm{ev}_0{}^* F$ is a smooth isomorphism of principal $\mathrm{SO}(n)$-bundles.

We now fix a polarisation $L^2(S^1, \mathbb{C}^n) = V_+ \oplus V_0 \oplus V_-$ by the projections on the positive, zero and negative part of the spectrum of $\frac{d}{d\theta}$. We have $V_0 \simeq \mathbb{C}^n$ (the constant sections), $V_+ = \mathrm{Span}\{z^n \vec{e}_i\,;\, n \in \mathbb{N}^{>0}\}$ and $V_- = \overline{V_+}$. We fix a $*$-representation of $\mathrm{Cl}(L^2(S^1,\mathbb{C}^n)) \simeq \mathrm{Cl}(V_0) \widehat{\otimes} \mathrm{Cl}(V_+ \oplus V_-)$ on

$$\mathcal{F}_0 := \mathbb{S} \,\hat{\otimes}\, \overline{\bigwedge V_+}$$

where $\mathbb{S}$ is the spinor representation for the finite dimensional Clifford algebra $\mathrm{Cl}(V_0)$ and the Hilbert closure of $\bigwedge V_+ \simeq \mathrm{Cl}_0(V_+ \oplus V_-)/\mathrm{Cl}_0(V_+ \oplus V_-) \cdot V_-$ is the faithful representation that arises as the second quantisation of the *complex* Hilbert space $L^2(S^1, \mathbb{R}^n)/V_0$ with complex structure induced by the S^1-action. The $\widehat{\otimes}$-sign denotes the $\mathbb{Z}_2$-graded tensor product for superalgebras and their representations.

This representation allows us to equip the algebras $\mathrm{Cl}(L^2(I, \mathbb{R}^n))$ with the strong topology of pointwise convergence on $\mathcal{F}_0$, and we denote its strong closure (which is the same as its double commutant) by $\mathrm{Cl}(I, \mathbb{R}^n)'' \subseteq B(\mathcal{F}_0)$.

We now define $M(I) := \mathrm{ev}_0^* F \times_{\mathrm{SO}(n)} \mathrm{Cl}(L^2(I, \mathbb{R}^n))''$. This is well defined because $\mathrm{SO}(n)$ has a projective unitary representation $\tilde{g} \mapsto U_{\tilde{g}}$ on $\mathcal{F}_0$, and the transition maps $\alpha_g \in \mathrm{Aut}(\mathrm{Cl}(L^2(I, \mathbb{R}^n)))$ can be written $\alpha_g(a) = U_{\tilde{g}} a U_{\tilde{g}}$ (the expression not depending on the lift $\tilde{g}$ of g to $\mathrm{Spin}(n)$), hence extend to the strong closure. The pullback of the strong topology under the isomorphism $M_\phi(I) \to \mathrm{Cl}(L^2(I, \mathbb{R}^n))''\colon [f_0, M] \mapsto M$ does not depend on the choice of initial frame $f_0 \in \mathrm{ev}_0^* F_\phi$, so each fibre $M_\phi(I)$ is a W^*-algebra.

Since both the inclusion $\iota\colon \mathcal{A}_\phi(I) \hookrightarrow \mathcal{A}_\phi(J)$ and the twist by the holonomy $\alpha_h\colon \mathcal{A}_\phi(I) \to \mathcal{A}(I)$ are strongly continuous, the transition maps $\mathcal{A}_\phi(I) \to \mathcal{A}_\phi(J)$ extend to morphisms $M_\phi(I) \to M_\phi(J)$, thus making $I \mapsto M_\phi(I)$ into a net of W^*-algebras.

If we choose a loop $\phi \in LM$ *and* an initial frame $f_0 \in \mathrm{ev}_0^* F_\phi$, then we obtain a $\mathrm{Fer}^n(S^1) - \mathrm{Fer}^n(S^1)$ defect $D_{\phi, f_0}\colon \mathrm{Int} \to \mathrm{vNAlg}$ if we use f_0 to identify $M_\phi(I) \simeq \mathrm{Cl}(L^2(I, \mathbb{R}^n))''$, and then compare the 'twisted' inclusion ι with the 'bare' inclusion ι^0. The outcome, of course, is the defect $D_{h(\phi, f_0)}$ described in Definition 4).

[**Show that the topology on $\mathrm{Cl}_\phi(I)$ induced by the $\mathbb{R}$-action corresponds to the topology on $\mathcal{A}_\phi(I)$ we've chosen, even though this is not the case for $I = S^1$!**]

7.5 Fusion

We will need spin structures with an additional structure called *fusion*, see [Wal13]. If X is a connected smooth manifold, we denote by PX the manifold of 'paths with sitting instants', i.e. smooth maps $\gamma\colon [0,1] \to X$ that are constant in a neighbourhood of $\{0, 1\}$ and we denote by

$$P^{[k]}X := \{(\gamma_1, \ldots, \gamma_k) \in (PX)^k \,;\, \gamma_i(0) = \gamma_j(0)\,,\, \gamma_i(1) = \gamma_j(1)\, \forall\, 1 \le i \le j \le k\}$$

the manifold of k-tuples of paths that start and end at the same point. Because of the sitting instants, the loops $\gamma_{ij} := \overline{\gamma}_j * \gamma_i$ are smooth, $\gamma_{ij} \in LM$.

For every $U(1)$-bundle $P \to LX$ over LX, a *fusion structure* assigns to each $(\gamma_1, \gamma_2, \gamma_3) \in P^{[3]}X$ a smooth $U(1)$-equivariant map $\lambda_{\gamma_1, \gamma_2, \gamma_3}\colon P_{\gamma_{12}} \otimes P_{\gamma_{23}} \to P_{\gamma_{13}}$ that is associative

$$\lambda_{\gamma_1, \gamma_3, \gamma_4}(\lambda_{\gamma_1, \gamma_2, \gamma_3}(p_{12} \otimes p_{23}) \otimes p_{34}) = \lambda_{\gamma_1, \gamma_2, \gamma_4}(p_{12} \otimes \lambda_{\gamma_2, \gamma_3, \gamma_4}(p_{23} \otimes p_{34}))$$

and smooth in the sense that if U is any manifold and $c\colon U \to P^{[3]}X$ is such that all 3 projections $e_{ij}\colon (\gamma_1, \gamma_2, \gamma_3) \mapsto \gamma_{ij}$ to LX are smooth, then $\lambda_c\colon e_{12}^*P \otimes e_{23}^*P \to e_{13}^*P$ is a smooth bundle morphism over U.

Theorem 28. *The central $U(1)$-extension*

8 Questions

A number of questions that pop up in relation to the above. I have not yet allotted much effort to answering them, so I do not know which, if any, of these are hard.

Homotopy groups

First a terribly naive question: in [DH12], it is shown that the geometric realisation of $G(V)$ (for $\dim(d) \geq 5$) is the 3-connected cover of $O(V)$. Is there a relation between the topology of the geometric realisation and the topology of the continuous model Γ?

Dimension 4

The Lie group $SO(4)$ is, in contrast to $SO(d)$ with $d \geq 5$, not a simple Lie group; $\mathrm{Spin}(4) = \mathrm{Spin}(3) \times \mathrm{Spin}(3)$. This allows you to write the algebras generated by the gauge group as a product of independent factors. How does this affect the structure of the corresponding 2-group?

Which unitary group?

In the construction of the strict model for $G(V)$, we've used the group of even unitary operators that are generated by fields on the right hand side of the circle, $\mathcal{A}_{\mathrm{ev}(S^1_R)}$. The reason for this (apart from the fact that you get to the Stolz-Teichner model) is mainly a matter of convenience: if $U, V \in \mathcal{A}_{\mathrm{ev}(S^1_R)}$, then they commute with x_0 so that

$$UxV\xi = Ux_0U_{h,R}^{-1}V\xi = Ux_0U_{h,R}^{-1}VU_{h,R}U_{h,R}^{-1}\xi = U\alpha_{h^{-1}}(V)x(\xi)$$

and we get an easy expression for the unitary operator $x(\xi) \mapsto Ux(V\xi)$. But this is just a luxury. I don't see why the map $x\xi \mapsto xV\xi$ could not also be well defined for other unitary operators V, perhaps even for all unitary operators if we do not adhere to the natural transformation $x \otimes \xi \mapsto x(\xi)$. We might get a unitary group that is a bit bigger.

On the other hand, a smaller group could also do the trick. To be precise, Stolz and Teichner use in their construction the algebra of unitary operators generated by the loop group representation, not the even operators generated by the fermionic fields, which is a priori smaller. (for semisimple gauge groups, i.e. dimension > 4, they should be the same though). It's not a big difference, but it goes to show that there seems to be some leeway in the type of unitary group that you choose here. Which is the best one? And what are good criteria to decide this?

Continuous/nice string groups

One gets the impression that the isomorphism $\Gamma \simeq G(V)$ (or something like it) is somehow 'nicer' than just an isomorphism of weak 2-groups. Is it possible to construct a canonical inverse for each sector (by reflecting it), and then show that the isomorphism respects these inverses? Is it possible to restrict $G_0(V)$ a bit so that $G(V)$ becomes a small category, perhaps even internal in the category of topological spaces? One idea would be to fix a Riemannian manifold M, and then take $G_0(V)$ to be the sectors that come from loops in M. The space of loops has a topology, and for any two homotopic loops, parallel transport along the the Levi-Civita connection identifies the corresponding Fock spaces (up to $U(1)$ because the transport operator probably doesn't have a canonical quantisation?), which may make it possible to decide whether the corresponding sectors are 'close'.

Smooth string groups

The Stolz-Teichner model is continuous, but not smooth. The problem is that the unitary group, equipped with the strong topology, is not an infinite dimensional Lie group. It *is* a Lie group (modeled on the Banach space of bounded hermitean operators) w.r.t. the norm topology, but swiching to the norm topology won't work because all positive energy representations of the Lie algebra $\Omega\mathfrak{so}(V)$ are by unbounded operators. Besides, it seems to me a somewhat counterintuitive thing to do in the context of v.N. algebras.

We have some freedom in our choice of unitary group though. Is it possible to choose a group of unitary operators that is contractible (we need that to get the right topology), big enough to contain the image of the gauge group, yet small enough to allow for a smooth structure?

Type I_∞ or type III_1?

If I understand the paper [CR87] by Carey and Ruissenaars correctly (which is definitely open to debate), they claim that for *Dirac* fermions, which have Fock a space $\mathcal{F} \otimes \overline{\mathcal{F}}$ *with* antiparticles and which we are *not* considering in these notes (because $\mathcal{F} \otimes \overline{\mathcal{F}}$ is not the analogue of the spin representation), the operator algebras generated by the gauge group $\Omega U(n)$ on subsets of the circle are hyperfinite type III_1 factors. On the other hand, for *Majorana* fermions, which have Fock a space $\mathcal{F}$ *without* antiparticles and which we *are* considering in these notes, the operator algebras generated by the gauge group $\Omega O(n)$ on subsets of the circle are supposedly your garden variety type I_∞ factors. This would be rather nice, because it would mean that in order to prove that our string group has the appropriate topological properties, we need not rely on the fact (?) that the unitary group of a type III_1 factor is contractible. The fact that $U(\mathcal{H})$ is contractible in the weak topology would suffice.

One thing that confuses me is that if I understand Stolz and Teichner [ST04] correctly (which is open to almost unimaginable amounts of debate), they work with Majorana fermions and also claim to encounter type III_1 factors.

The big question

If (M, p) is a pointed riemannian manifold, then every parameterised closed loop $L : S^1 \to M$ starting at p carries a vector bundle L^*TM with a covariantly flat section (w.r.t. the pull-back of the Levi-Civita connection). The bundle L^*TM is either trivial or (if M is nonorientable of odd dimension) the Moebius band. This yields a bundle of C^*-algebras $\mathrm{CAR}(\Gamma(L^*TM \downarrow S^1))$ over L_pM that reminds one of a bundle of Clifford algebras.

If we want our analogue of the Clifford bundle to be a bundle of von Neumann algebras, then the natural states w.r.t. which to complete the CAR-algebra would presumably be the vacuum states w.r.t. the polarisation into positive and negative parts of the spectrum of the differential operator $i\nabla_\phi$. In view of prop. 12 (which is really the thing on which all the above hinges), you'd expect these states to be normal w.r.t. each other if and only if the holonomy of the loops is the same. So each fibre of the holonomy map $L_pM \to O(T_pM)$ would carry something that looks like a bundle of v.N. algebras (namely the ones corresponding to the defect given by the holonomy), but there's no bundle on all of L_pM because the algebras at different points are not isomorphic.

If 2 loops have a segment in common, it seems that one can take the fusion product of the corresponding field theories to get a field theory that corresponds to the holonomy of the fused loops. What kind of structure do you get? Does the string 2-group act on it as if it were some kind of 2-bundle with a fusion product? I have not read all of [ST04], perhaps the answer is in there.

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