HOW TO THINK OF QUANTUM MARKOV MODELS FROM AN ENGINEERING PERSPECTIVE

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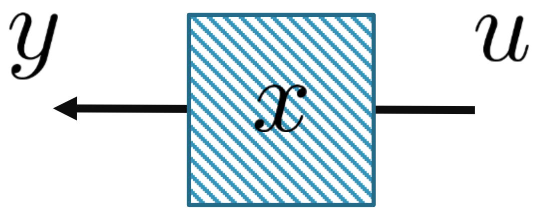
"Mathematics of QIT"

May 6-10 Lorentz Center

Input-Plant-Output Models

- Plant = System (state variable *x*)



$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

- Laplace domain

$$Y(s) = T(s)\,U(s)$$

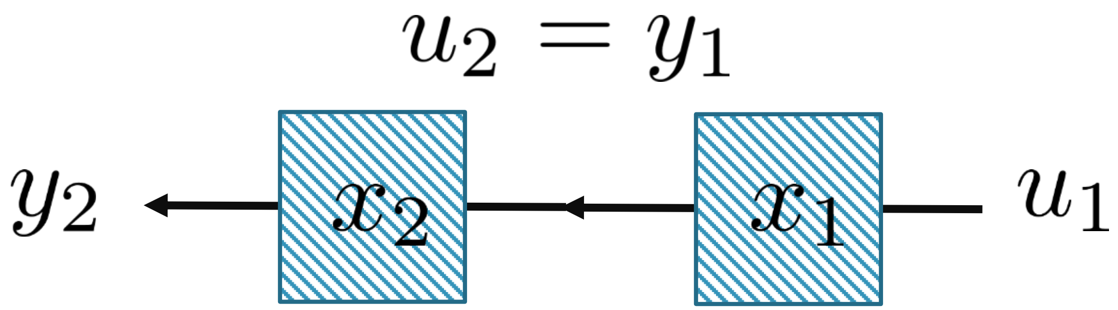
- Transfer Function

$$T(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right](s) = D + C\frac{1}{A-s}B.$$

Block Diagrams

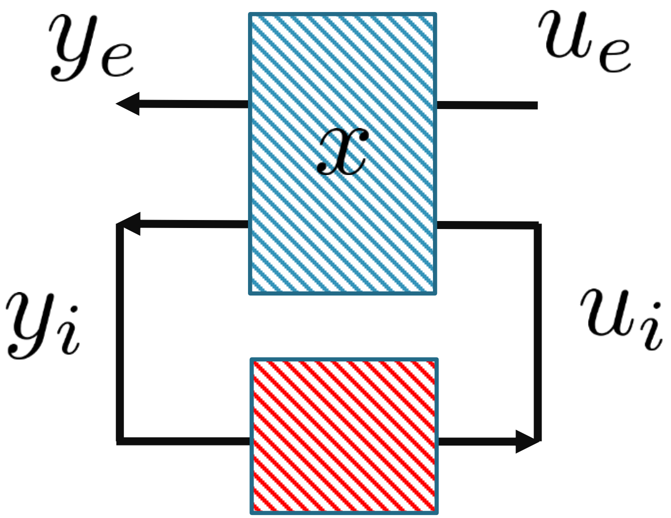
- Series

$$T(s) = T_2(s)\,T_1(s).$$



- Feedback

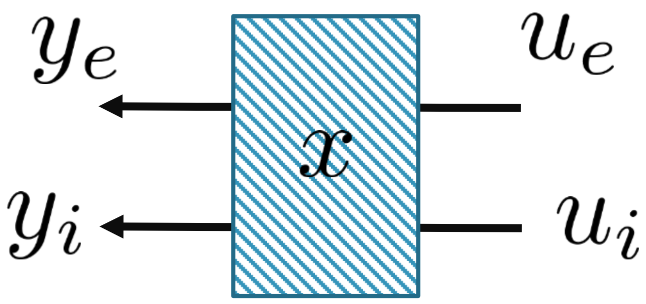
$$u_i = K\,y_i$$



Fractional Linear Transformations

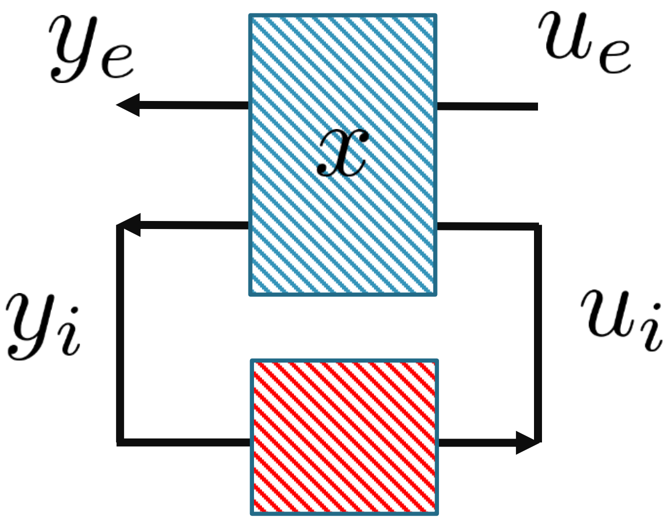
- "Open Loop"

$$
\begin{aligned}
\dot{x} &= Ax + B_e u_e + B_i u_i \\
y_e &= C_e x + D_{ee} u_e + D_{ei} u_i \\
y_i &= C_i x + D_{ie} u_e + D_{ii} u_i.
\end{aligned}
$$



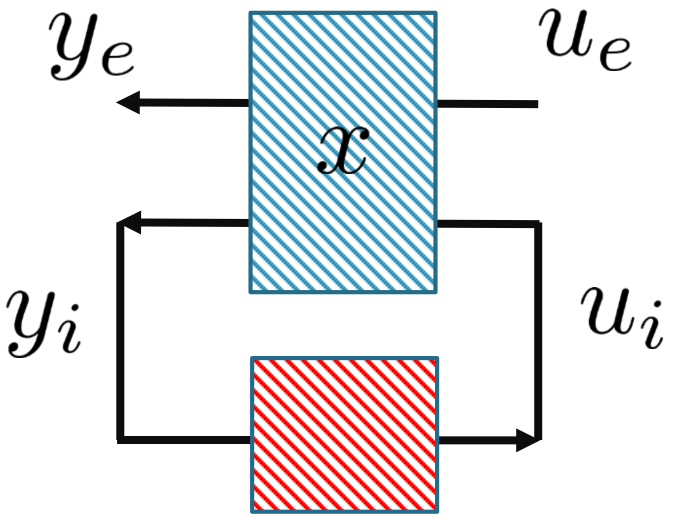
- "Closed Loop"

$$
\begin{aligned}
\dot{x} &= \hat{A}x + \hat{B}_e u_e \\
y_e &= \hat{C}_e x + \hat{D}_{ee} u_e.
\end{aligned}
$$



Fractional Linear Transformation

- The feedback reduction



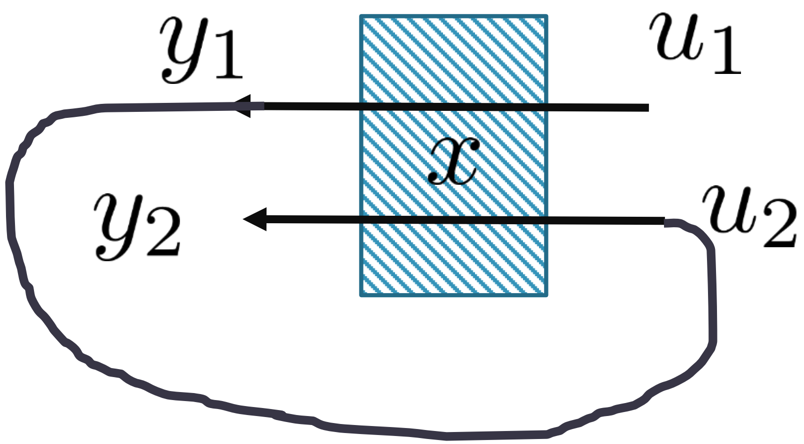
$$\begin{bmatrix} A & B_e & \boxed{B_i} \\ C_e & D_{ee} & D_{ei} \\ C_i & D_{ie} & D_{ii} \end{bmatrix}$$

$$\mapsto \begin{bmatrix} \hat{A} & \hat{B}_e \\ \hat{C}_e & \hat{D}_{ee} \end{bmatrix} = \begin{bmatrix} A & B_e \\ C_e & D_{ee} \end{bmatrix} + \begin{bmatrix} \boxed{\begin{matrix} B_i \\ D_{ei} \end{matrix}} \end{bmatrix} (1 - K\boxed{D_{ii}})^{-1} K \left[\boxed{C_i \quad D_{ie}} \right].$$

- Algebraic loops if $D_{ii} \neq 0$

Double Pass!

- Special case of feedback reduction



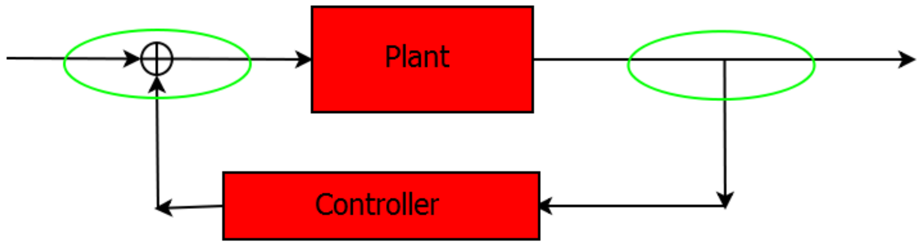
$$\left[\begin{array}{cc|c} A & B_1 & B_2 \\ C_1 & D_1 & 0 \\ C_2 & 0 & D_2 \end{array}\right]$$

$$\mapsto \begin{bmatrix} A & B_1 \\ C_2 & 0 \end{bmatrix} + \begin{bmatrix} \boxed{\begin{matrix} B_2 \\ D_2 \end{matrix}} \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} C_1 & D_1 \end{matrix}} \end{bmatrix} = \begin{bmatrix} A + B_2 C_1 & B_1 + B_2 D_1 \\ C_2 + D_2 C_1 & D_2 D_1 \end{bmatrix}$$

$$(A \equiv A_1 + A_2)$$

$$=: \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \lhd \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix}.$$

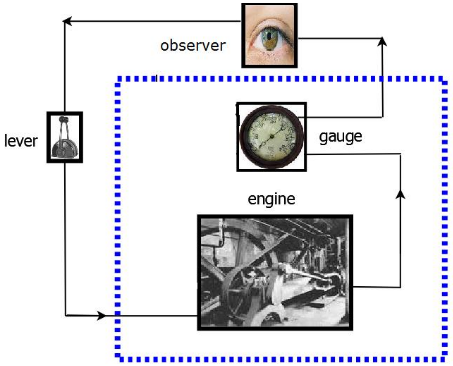
Networks and Feedback Control



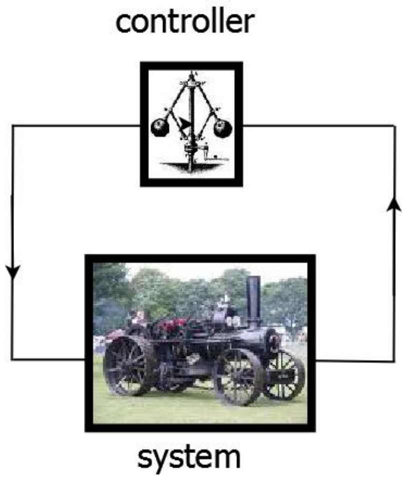
cannot happen in the quantum setting!!!

must use unitary junctions (e.g., beamsplitters)

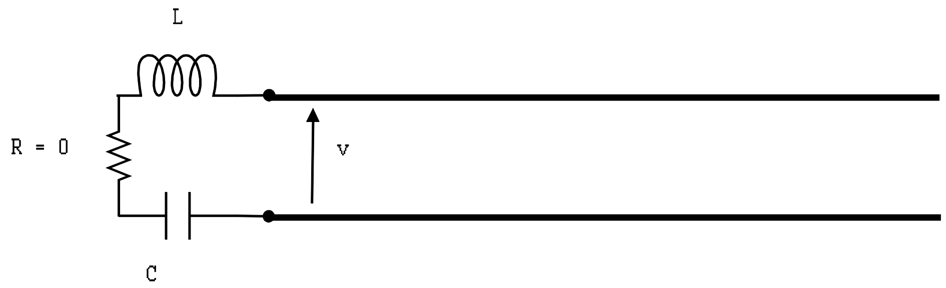
- Measurement Based Feedback Control

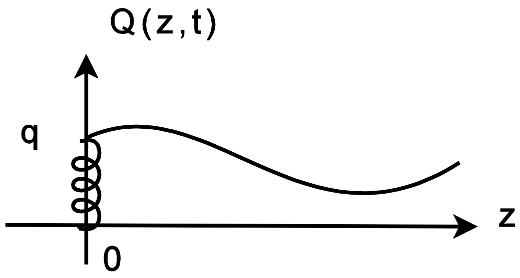


- Coherent Feedback Control



Quantum Inputs and Outputs





Lamb Model / Caldeira-Leggett / Ford-Kac-Mazur / Thirring-Schwabl /Lewis-Maassen/ Yurke-Denker

Terminal charge $q(t) = Q(z = 0, t)$

$L\ddot{q}(t) + R\dot{q}(t) + \frac{1}{C}q(t) = F(t)$, with $R = \frac{1}{\kappa c}$ and $F(t) = 2R\dot{q}_{\text{in}}(t)$

$$\boxed{[F(t), F(s)] = i\frac{d}{dt}\delta(t-s)}$$

Non-Markov Models and Markov Limits

$$L\ddot{q}(t) + \int_{-\infty}^{t} \Gamma(t-t')\dot{q}(t') + \frac{1}{C}q(t) = F(t)$$

$$S[s] = \frac{Ls^2 - s\mathscr{L}\Gamma[s] + \frac{1}{C}}{Ls^2 + s\mathscr{L}\Gamma[s] + \frac{1}{C}}$$

$\mathscr{L}\Gamma[s]$ - Linear Positive Real,

$S[s]$ - Linear Bounded Real

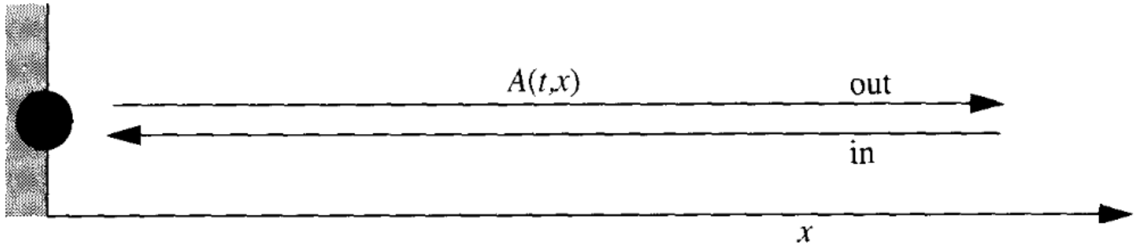
Input-output relations

$$\hat{q}_{\mathrm{out}}(\omega) = S[0^+ - i\omega]\,\hat{q}_{\mathrm{in}}(\omega)$$

Spectral Density

$$J(\omega) = \frac{1}{\omega}\mathrm{Re}\mathscr{L}\Gamma[0^+ - i\omega].$$

Gardiner-Collett



Quantum Markovian Dynamics

- A semi-group of CP identity-preserving maps (Heisenberg picture!)
- Generator (Lindblad) $\Phi_t \equiv e^{t\mathscr{L}}$,

$$\boxed{\mathscr{L}X \equiv \mathscr{L}_{\mathbf{L},H}X = \frac{1}{2}\sum_{k=1}^{d}[L_k^*, X]L_k + \frac{1}{2}\sum_{k=1}^{d}L_k^*[X, L_k] - i[X, H].}$$

- Dilation

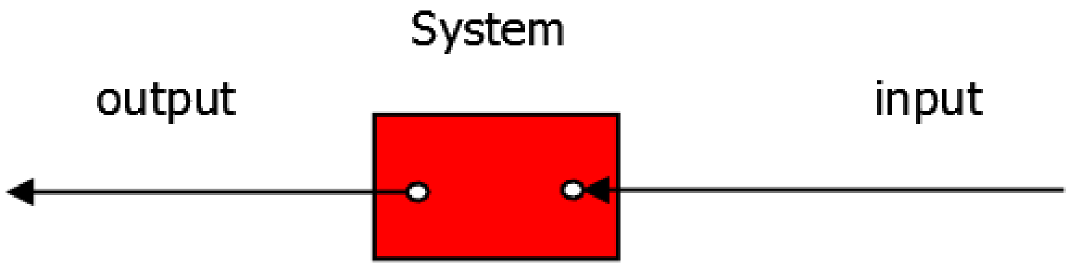
$$\Phi_t(X) = \mathrm{tr}_{\mathfrak{F}}\left\{\mathbb{1} \otimes |\Omega\rangle\langle\Omega| \ U(t)^*[X \otimes \mathbb{1}]U(t)\right\}$$

auxiliary space $\mathfrak{F}$, vector state $|\Omega\rangle \in \mathfrak{F}$, unitary $U(t)$ on $\mathfrak{h}_{\mathrm{sys}} \otimes \mathfrak{F}$

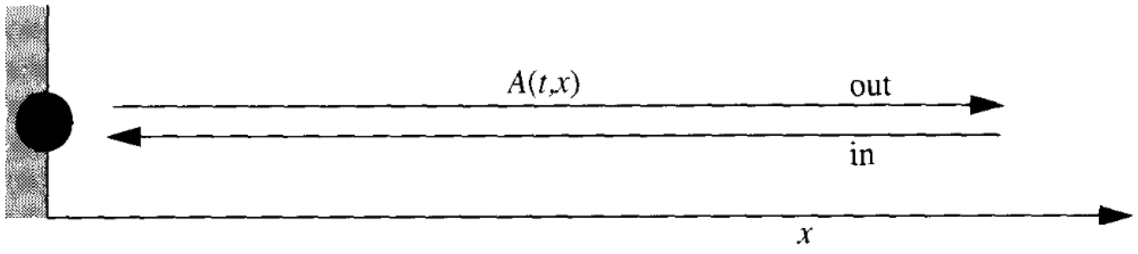
Quantum Input-Output Systems

Hudson-Parthasarathy (1984)

V.P. Belavkin (1979+)

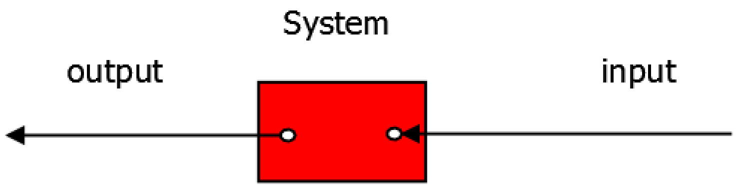


Gardiner-Collett (1985)



Quantum Input Processes

The "wires" are quantum fields!



- Field quanta of type *k* annihilated at the system at time *t*: $b_{\mathrm{in},k}(t)$

$$[b_{\mathrm{in},j}(t), b_{\mathrm{in},k}(s)^*] = \delta_{jk}\,\delta(t-s).$$

- Hilbert Space: $\mathfrak{F} = \bigoplus_{n=0}^{\infty} \left(\otimes^n_{\mathrm{symm.}} \mathfrak{h}_{1\mathrm{P}} \right), \qquad \mathfrak{h}_{1\mathrm{P}} = \oplus_k L^2[0,\infty).$

- Default state is the (Fock) vacuum $|\Omega\rangle$

$$b_{\mathrm{in},k}(t)\,|\Omega\rangle \equiv 0.$$

Quantum Stochastic Models

Single input – Emission/Absorption Interaction

$$\dot{U}(t) = \left\{ L \otimes b^*(t) - L^* \otimes b(t) - iH \otimes I \right\} U(t)$$

- Wick-ordered form:

$$\dot{U}(t) = L \otimes b^*(t) U(t) - L^* \otimes U(t) b(t) - (\tfrac{1}{2} L^* L + iH) \otimes I \, U(t)$$

- Heisenberg Picture $\quad j_t(X) = U(t)^* (X \otimes I) U(t)$

$$\boxed{\frac{d}{dt} j_t(X) = b^*(t) j_t([L, X]) + j_t([X, L^*]) b(t) + j_t(\mathscr{L} X)}$$

- GKS-Lindblad Generator

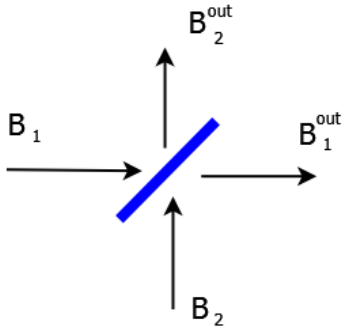
$$\mathscr{L} X = \tfrac{1}{2} L^* [X, L] + \tfrac{1}{2} [L^*, X] L - i[X, H]$$

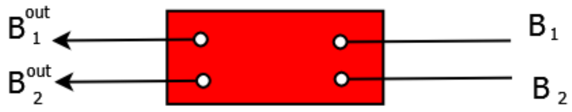
- Input-Output Relations

$$\boxed{b_{\text{out}}(t) = I \otimes b(t) + j_t(L)}$$

Quantum Stochastic Models

- Two inputs – pure scattering





$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}, \qquad S^{-1} = S^*$$

Wick-ordered form: $\dot{U}(t) = \sum_{j,k} (S_{jk} - \delta_{jk}) \otimes b_j^*(t) U(t) b_k(t)$

Heisenberg Picture $\boxed{\frac{d}{dt} j_t(X) = \sum_{j,k} b_j^*(t) j_t\left(\sum_l S_{lk}^* X S_{lk} - \delta_{jk} X \right) b_k(t)}$

Input-Output Relations $\boxed{b_{\text{out},j}(t) = \sum_k j_t(S_{jk})\, b_k(t)}$

Quantum Ito Table

- Fundamental Processes

$$B_k(t) \equiv \int_0^t b_k(s)ds \qquad \Lambda_{jk}(t) \equiv \int_0^t b_j(s)^* b_k(s)ds$$

- Table

$$dB_j dB_k^* = \delta_{jk} dt$$

$$\begin{aligned} d\Lambda_{jl} dB_k^* &= \delta_{lk} dB_j^* \\ dB_j d\Lambda_{kl} &= \delta_{jk} dB_l \\ d\Lambda_{jl} d\Lambda_{ki} &= \delta_{lk} d\Lambda_{ji} \end{aligned}$$

- Product Rule

$$d(XY) = dX(t)\,Y(t) + X(t)\,dY(t) + dX(t)\,dY(t).$$

SLH Formalism

- Hamiltonian *H*

$$H^* = H$$

- Coupling/Collapse Operators *L*

$$L = \begin{bmatrix} L_1 \\ \vdots \\ L_n \end{bmatrix}$$

- Scattering Operator *S*

$$S = \begin{bmatrix} S_{11} & \cdots & S_{1n} \\ \vdots & \ddots & \vdots \\ S_{n1} & \cdots & S_{nn} \end{bmatrix}, \qquad S^{-1} = S^*$$

Quantum Stochastic Models

- General (S , L , H) case

Wick-ordered form:

$$\dot{U}(t) = b_j^*(t)(S_{jk} - \delta_{jk} I) U(t) b_k(t) + L_j \otimes b_j^*(t) U(t) - L_j^* S_{jk} \otimes U(t) b_k(t) - (\frac{1}{2} L_k^* L_k + iH) \otimes I \, U(t)$$

Or better as a QSDE (quantum Ito stochastic calculus)

$$\boxed{dU(t) = \Big\{ (S_{jk} - \delta_{jk} I) \, d\Lambda_{jk}(t) + L_j \otimes dB_j^*(t) - L_j^* S_{jk} \otimes dB_k(t) - (\frac{1}{2} L_k^* L_k + iH) \otimes dt \Big\} U(t)}$$

Quantum Stochastic Models

Heisenberg Picture

$$j_t(X) = U(t)^*(X \otimes I)U(t)$$

$$\boxed{\begin{aligned} dj_t(X) \quad &= \quad j_t(S_{lj}^* X S_{lk} - \delta_{jk} X) d\Lambda_{jk}(t) + j_t(S_{lj}^*[L_l, X]) dB_{\text{in},j}^*(t) \\ &\quad + j_t([X, L_l^*] S_{lk}) dB_{\text{in},k(t)} + j_t(\mathscr{L} X) dt. \end{aligned}}$$

Lindblad Generator

$$\mathscr{L} X = \tfrac{1}{2} L_k^* [X, L_k] + \tfrac{1}{2} [L_k^*, X] L_k - i[X, H]$$

Input-Output Relations

$$B_{\text{out},k}(t) = U(t)^*(I \otimes B_{\text{in},k}(t))U(t)$$

$$\boxed{dB_{\text{out},j}(t) = j_t(S_{jk}) dB_{\text{in},k}(t) + j_t(L_j) dt}$$

This is Markovian!

"Pyramidal" Multi-time Expectations $(t_n > \cdots > t_1)$

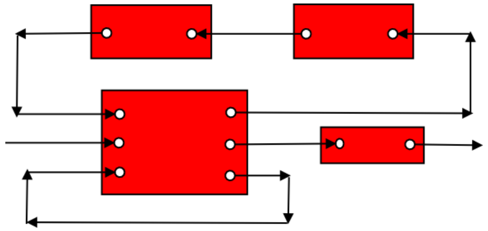
$$tr\left\{\rho_0 \otimes |\Omega\rangle\langle\Omega|\; j_{t_1}(A_1^\dagger)\cdots j_{t_n}(A_n^\dagger) j_{t_n}(B_n)\cdots j_{t_1}(B_1)\right\}$$

$$= tr\left\{\rho_0 \Phi_{t_1}\left(A_1^\dagger \Phi_{t_2-t_1}\left(A_2^\dagger \cdots \Phi_{t_n-t_{n-1}}\left(A_n^\dagger B_n\right)\cdots\right) B_1\right)\right\}.$$

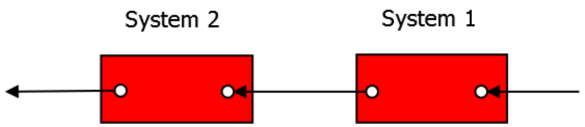
In non-Markovian models there is no "state" in the usual sense!

Quantum Networks

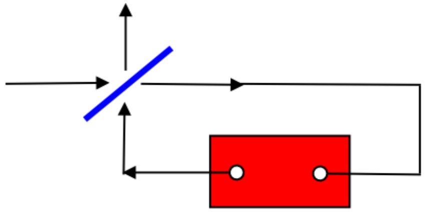
- How to connect models?



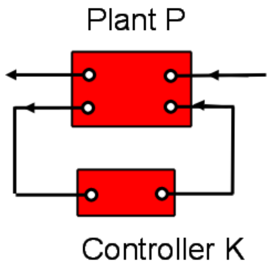
- Cascaded models



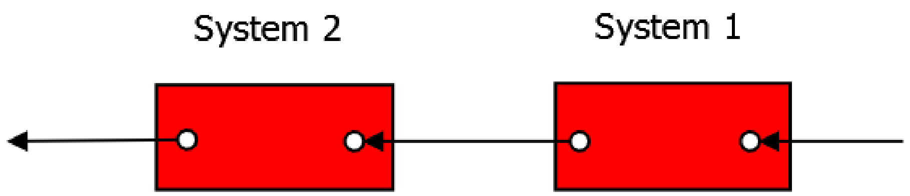
- Algebraic loops



- Feedback Control



The Series Product



The cascaded system in the **instantaneous feedforward** limit is equivalent to the single component

$$\boxed{(S_2, L_2, H_2) \triangleleft (S_1, L_1, H_1) = \left(S_2 S_1, L_2 + S_2 L_1, H_1 + H_2 + \mathrm{Im}\left\{L_2^\dagger S_2 L_1\right\}\right).}$$

J. G., M.R. James, *The Series Product and Its Application to Quantum Feedforward and Feedback Networks* IEEE Transactions on Automatic Control, 2009.

Perturbations (Avron, Fraas, Graf)

- Virtual displacement of the model $(S, L, H) \triangleleft (I, \delta L, \delta H)$

Displacements

$$X \to e^{iG\delta\phi} X e^{-iG\delta\phi} \qquad\qquad \delta X = i[X, G]\,\delta\phi.$$

Virtual work

$$\begin{aligned}\delta H + \frac{1}{2i}\Big(L^*\,\delta L - (\delta L)^*\,L\Big) &= \Big(-i[G,H] + \frac{1}{2}L^*[L,G] + \frac{1}{2}[L^*,G]L\Big)\,\delta\phi \\ &= \mathcal{L}_{L,H}(G)\,\delta\phi.\end{aligned}$$

Local Asymptotic Normality

- Suppose that the QMS has a unique faithful stationary state

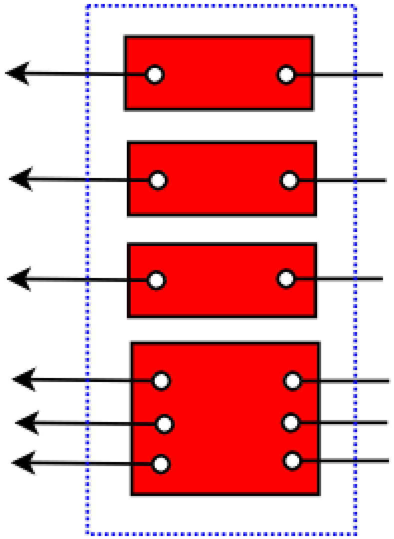
$$\text{Time-averages: } \mathbb{M}_T(X) = \tfrac{1}{T}\int_0^T j_t(X)\,dt \to \langle X \rangle\, I.$$

- (CCR) Algebra of fluctuations (Guta and Kiukas, Bouten)

$$\text{Fluctuations: } \mathbb{F}_T(X) = \tfrac{1}{\sqrt{T}}\int_0^T \big(j_t(X) - \langle X \rangle\big)\,dt.$$

- Geometric structure closely related to the Series Product!
- General perturbations (Bouten and JG)
- PROBLEM: is this some form of de Bruijn identity?

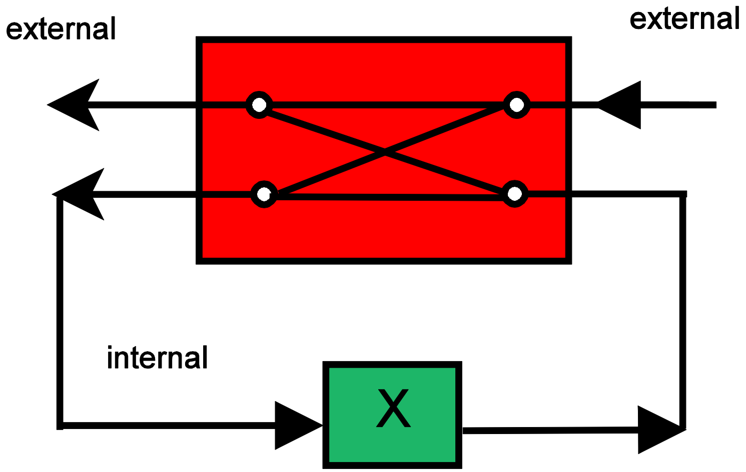
Network Rule # 1
Open loop systems in parallel



Models $(S_j, L_j, H_j)_{j=1}^{n}$ in parallel

$$\left(\begin{bmatrix} S_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & S_n \end{bmatrix}, \begin{bmatrix} L_1 \\ \vdots \\ L_n \end{bmatrix}, H_1 + \cdots + H_n \right).$$

Network Rule # 2
Feedback Reduction Formula



$$S = \begin{bmatrix} \mathsf{S}_{\mathtt{ii}} & \mathsf{S}_{\mathtt{ie}} \\ \mathsf{S}_{\mathtt{ei}} & \mathsf{S}_{\mathtt{ee}} \end{bmatrix}, \mathsf{L} = \begin{bmatrix} \mathsf{L}_{\mathtt{i}} \\ \mathsf{L}_{\mathtt{e}} \end{bmatrix}$$

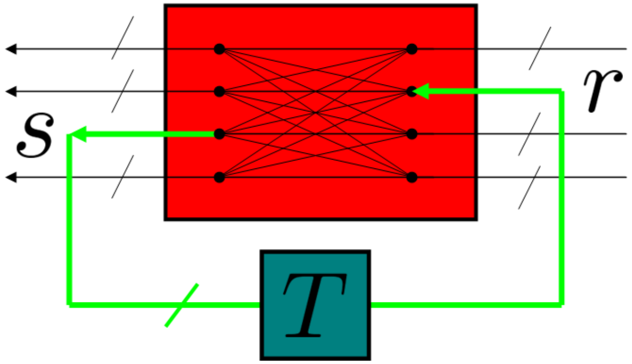
The reduced model obtained by eliminating all the internal channels (instantaneous feedback) is determined by the operators $(\mathsf{S}^{\mathrm{fb}}, \mathsf{L}^{\mathrm{fb}}, \mathsf{H}^{\mathrm{fb}})$ given by

$$\begin{aligned} \mathsf{S}^{\mathrm{fb}} &= \mathsf{S}_{\mathtt{ee}} + \mathsf{S}_{\mathtt{ei}} X \left(1 - \mathsf{S}_{\mathtt{ii}} X\right)^{-1} \mathsf{S}_{\mathtt{ie}}, \\ \mathsf{L}^{\mathrm{fb}} &= \mathsf{L}_{\mathtt{e}} + \mathsf{S}_{\mathtt{ei}} X \left(1 - \mathsf{S}_{\mathtt{ii}} X\right)^{-1} \mathsf{L}_{\mathtt{i}}, \\ \mathsf{H}^{\mathrm{fb}} &= \mathsf{H} + \sum_{i=\mathtt{i},\mathtt{e}} \mathrm{Im} \mathsf{L}_j^{\dagger} X \mathsf{S}_{j\mathtt{i}} \left(1 - \mathsf{S}_{\mathtt{ii}} X\right)^{-1} \mathsf{L}_{\mathtt{i}}. \end{aligned}$$

J. G., M.R. James, *Quantum Feedback Networks: Hamiltonian Formulation*, Commun. Math. Phys., 1109-1132, Volume 287, Number 3 / May, 2009.

$$\mathsf{V} = \begin{bmatrix} -\frac{1}{2}L^*L - iH & -L^*S \\ L & S \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}\sum_j L_j^* L_j - iH & -\sum_j L_j^* S_{j1} & \cdots & -\sum_j L_j^* S_{jm} \\ L_1 & S_{11} & \cdots & S_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ L_n & S_{n1} & \cdots & S_{nn} \end{bmatrix}$$

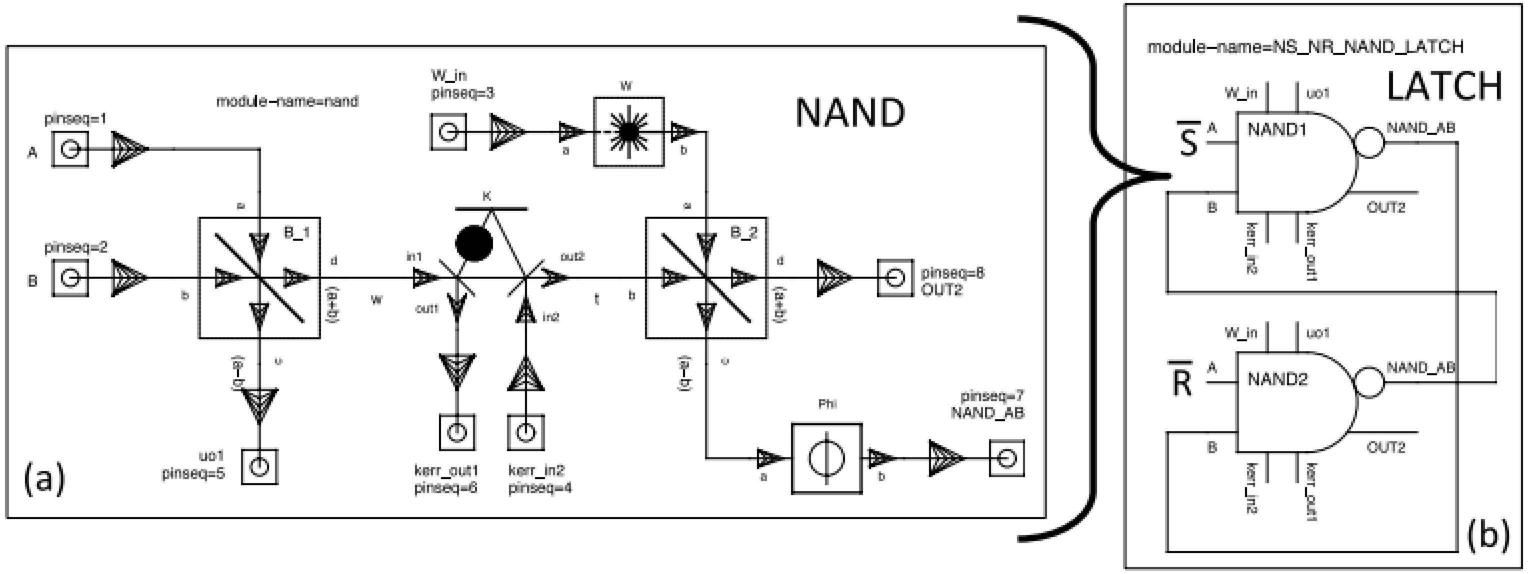
$$= \begin{bmatrix} \mathsf{V}_{00} & \mathsf{V}_{01} & \cdots & \mathsf{V}_{0m} \\ \mathsf{V}_{10} & \mathsf{V}_{11} & \cdots & \mathsf{V}_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ \mathsf{V}_{n0} & \mathsf{V}_{n1} & \cdots & \mathsf{V}_{nn} \end{bmatrix}.$$



The feedback reduction formula is

$$\left[\mathscr{F}_{(r,s)}(\mathsf{V}, T)\right]_{\alpha\beta} = \mathsf{V}_{\alpha\beta} - \mathsf{V}_{\alpha r} T \left(1 - \mathsf{V}_{rs} T\right)^{-1} \mathsf{V}_{s\beta}$$

The Network Rules are implemented in a workflow capture package QHDL



QHDL (MabuchiLab)

N. Tezak, et al., (2012) Phil. Trans. Roy. Soc. A, 370, 5270.

Transfer Operator

- Classical Transfer function

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right](s) = D + C\,(sI - A)^{-1}\,B,$$

- SLH version

For a fixed set of coupling parameters (S, L, H) we define the corresponding *transfer operator*

$$\boxed{\begin{aligned} \mathcal{T}(s) &= \left[\begin{array}{c||c} -\frac{1}{2}L^\dagger L - iH & -L^* S \\ \hline\hline L & S \end{array}\right](s) \\ &= S - L(sI + \frac{1}{2}L^* L + iH)^{-1} L^\dagger S \end{aligned}}$$

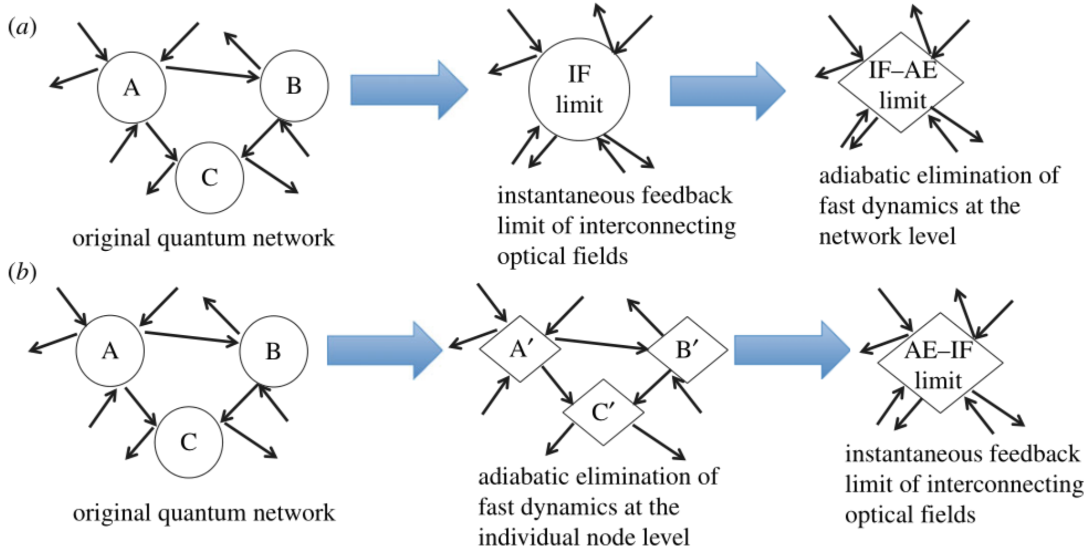
- Properties

The transfer operator $\mathcal{T}(s)$ is well-defined for $\mathrm{Re}\, s > 0$. For all $\omega \in \mathbb{R}$, such that $i\omega$ is not a "pole", we have $\mathcal{T}(\mathrm{i}\omega)$ well-defined and unitary:

$$\mathcal{T}(i\omega)^\dagger \mathcal{T}(i\omega) = \mathcal{T}(i\omega)\mathcal{T}(i\omega)^\dagger = I.$$

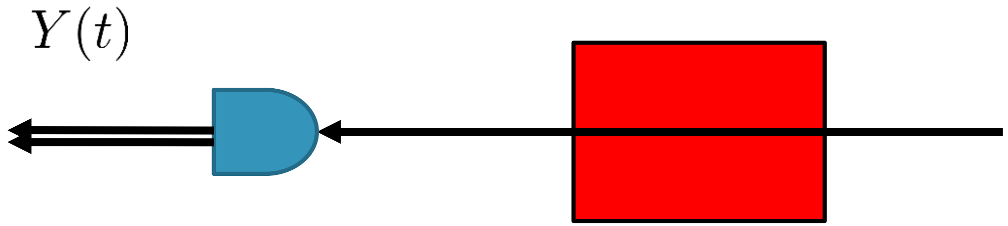
Adiabatic Elimination

- An important model simplification split the systems into slow and fast subspaces
- Mathematical this is also a fractional linear transformation



- It commutes with feedback reduction!
- JG, H. Nurdin, S. Wildfeuer, *J. Math. Phys., 51, 123518 (2010);* H. Nurdin, JG, *Phil. Trans. R. Soc., A 370, 5422-36 (2012)*

Measurement



- Homodyne

$$Z(t) = B_{\mathrm{in},k}(t) + B_{\mathrm{in},k}(t)^*$$

$$Y(t) = U(t)^*(I \otimes Z(t))U(t)$$

- Compatibility

$$[Y(s), Y(t)] = 0, \qquad \forall t, s \geq 0.$$

$$[j_t(X), Y(s)] = 0, \qquad t \geq s \geq 0.$$

- Filter

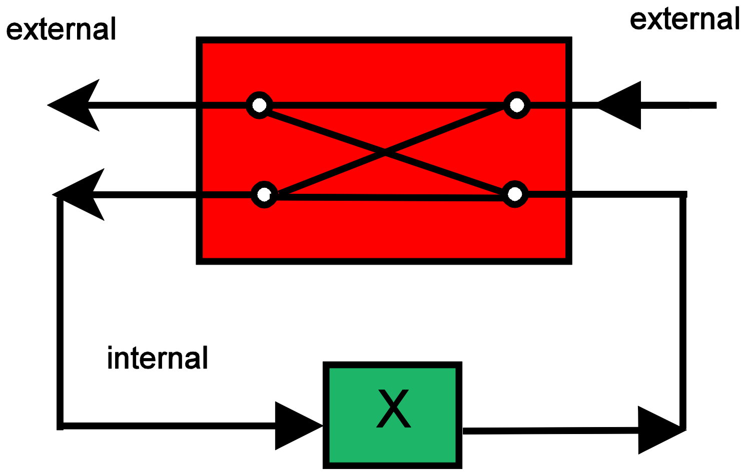
Conditioned state

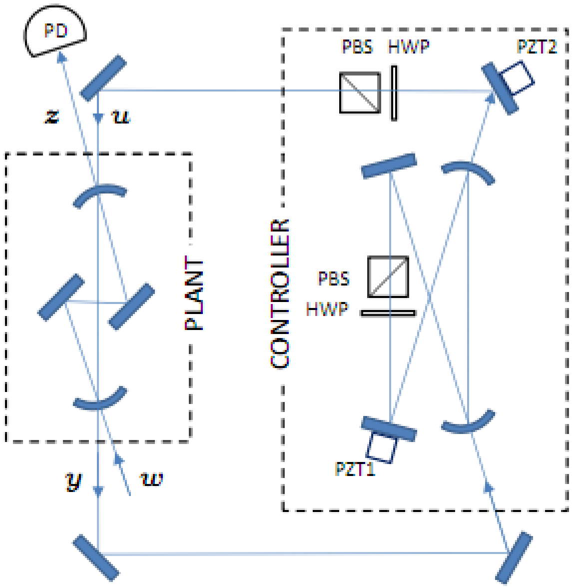
$$d\hat{\rho}_t = \mathscr{L}^\star \hat{\rho}_t \, dt + \Big(L\hat{\rho}_t + \hat{\rho}_t L^* - \mathrm{tr}\{\hat{\rho}_t (L + L^*)\} \Big) \, dI_t,$$

Innovations

$$dI_t = dY_t - \mathrm{tr}\{\hat{\rho}_t (L + L^*)\} \, dt.$$

Coherent Quantum Feedback Control





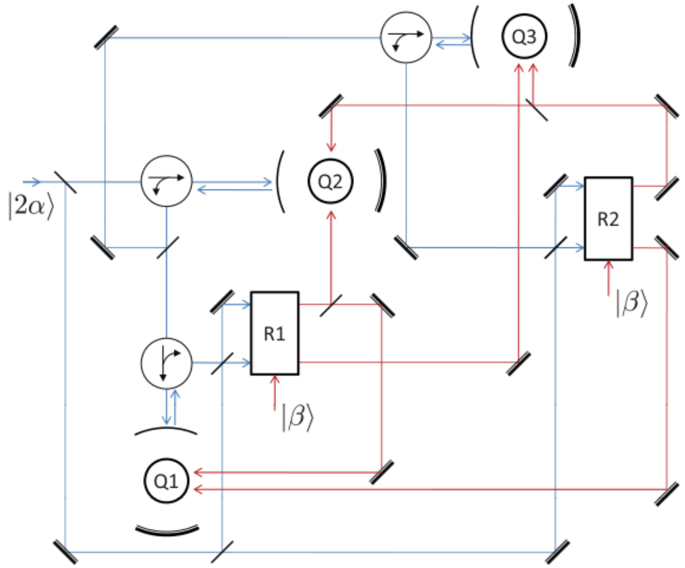
H. Mabuchi, *Coherent-Feedback Quantum Control With a Dynamic Compensator*, Phys. Rev. A 78, 032323 (2008).

Autonomous Quantum Error Correction

PRL 105, 040502 (2010) PHYSICAL REVIEW LETTERS week ending 23 JULY 2010

Designing Quantum Memories with Embedded Control: Photonic Circuits for Autonomous Quantum Error Correction

Joseph Kerckhoff,[1,*] Hendra I. Nurdin,[1,2,†] Dmitri S. Pavlichin,[1] and Hideo Mabuchi[1]



J. Kerckhoff, H. I. Nurdin, D. Pavlichin and H. Mabuchi, *Designing quantum memories with embedded control: photonic circuits for autonomous quantum error correction*, Phys. Rev. Lett. 105, 040502 (2010)

Thank You For Your Attention,