Loop groups

Bas Janssens

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In these notes, we introduce matrix Lie groups G and their Lie algebras $\mathrm{Lie}(G)$, and we exhibit the (continuous) loop group LG as a smooth Banach Lie group. Prerequisites are basic calculus and point set topology, but no knowledge of differential geometry is presupposed. For this reason, the presentation of the material is somewhat unconventional. Most importantly, the definition of matrix Lie groups and their Lie algebras is different from - but ultimately equivalent to - the usual definition in the literature [Hu72, DK00].

1 The loop space LM

Let (M, d) be a metric space. A *path* in M is a continuous map $\phi\colon [0,1] \to M$. A *loop* is a path which starts and ends in the same point, $\phi(0) = \phi(1)$. If we identify the endpoints of the interval $[0,1]$ with each other, we obtain the circle S^1. Since $\phi(0) = \phi(1)$, we can think of a loop as a continuous map of S^1 into M. As the name suggests, the *loop space* LM of M is the space of all loops,

$$LM = \{\phi\colon [0,1] \to M\,;\, \phi \text{ is continuous and } \phi(0) = \phi(1)\}\,.$$

The first aim is to have a closer look at LM.

2 Topology of loop spaces

First, we show that LM inherits from M the property of being a metric space. A *metric* on M is a map $d\colon M \times M \to \mathbb{R}^{\geq 0}$ with

$$d(x,y) = d(y,x) \qquad \text{(symmetry)} \tag{1}$$

$$d(x,z) \geq d(x,y) + d(y,z) \qquad \text{(triangle inequality)} \tag{2}$$

$$d(x,y) = 0 \Leftrightarrow x = y \qquad \text{(nondegeneracy)} \tag{3}$$

for all $x, y, z \in M$. We define the metric d_∞ on LM by

$$d_\infty(\phi, \chi) = \sup_{t \in [0,1]} d(\phi(t), \chi(t))\,.$$

The following exercise shows that we can consider LM as a topological space in its own right.

Exercise 1. Check that d_∞ is indeed a metric; show that the properties (1), (2), and (3) for d_∞ follow from the corresponding properties for d.

The loop space LM is sometimes called the *free* loop space, because a loop $\phi\colon [0,1] \to M$ can start and end wherever it wants to. For any $x \in M$, the *based* loop space $\Omega_x M \subseteq LM$ is the subspace of loops which start and end at the point x,

$$\Omega_x M = \{\phi \in LM\,;\, \phi(0) = \pi(1) = x\}\,.$$

Another interesting subspace of LM is the space of *constant* loops.

Exercise 2. Consider the map $M \to LM$ that takes a point x to the constant loop $\phi(t) = x$. Show that this is an isometry of M onto the space of constant maps.

In the following, we will therefore identify $M \subseteq LM$ with the space of constant maps.

Exercise 3. Show that the map $\mathrm{ev}_0\colon LM \to M$ with $\mathrm{ev}_0(\phi) = \phi(0)$ is continuous, surjective, and that the preimage of x is the based loop space $\mathrm{ev}_0^{-1}(x) = \Omega_x M$. Use this to show that $\Omega_x M$ is a closed subspace of LM.

3 Examples: $L\mathbb{R}$ and LS^1

We can ask ourselves what the 'shape' of LM is. For example, is it connected? Path connected? Simply connected? The answer, of course, depends rather crucially on M.

Consider, for example, the case $M = \mathbb{R}$. The loop space $L\mathbb{R}$ consists of continuous functions $\phi\colon S^1 \to \mathbb{R}$.

Exercise 4. Show that $L\mathbb{R}$ and $\Omega_0\mathbb{R}$ are contractible (and hence in particular connected and simply connected).

Already more interesting is the case $M = S^1$. We consider $S^1 \simeq \mathbb{R}/\mathbb{Z}$, and equip S^1 with the quotient metric $d(\theta, \theta') = \min\{|\theta - \theta' - k|\,;\, k \in \mathbb{Z}\}$. (The shortest distance along the circle.) It turns out that LS^1 is neither connected nor simply connected.

Exercise 5. Show that LS^1 is homeomorphic to $S^1 \times \Omega_0 S^1$. (Hint: to find a homeomorphism $LS^1 \to S^1 \times \Omega_0 S^1$, decompose $\phi \in LM$ into a constant loop and a based loop.)

The *winding number* of a loop $\phi \in \Omega_0 S^1$ is the number of times it winds around the circle. More formally, let $\psi\colon [0,1] \to \mathbb{R}$ be the unique continuous lift of $\phi\colon [0,1] \to \mathbb{R}/\mathbb{Z}$ with $\psi(0) = 0$ and $\phi(t) = \psi(t) \bmod \mathbb{Z}$. Since $\phi(1) = 0$ in $S^1 \simeq \mathbb{R}/\mathbb{Z}$, we have $\psi(1) \in \mathbb{Z}$. We thus obtain a bijection

$$\Omega_0 S^1 \simeq \{\psi\colon [0,1] \to \mathbb{R}\,;\, \psi \text{ is continuous, } \psi(0) = 0, \text{ and } \psi(1) \in \mathbb{Z}\}.$$

Then the winding number of ϕ is defined as $k = \psi(1)$.

Exercise 6. Show that $d_\infty(\phi, \phi') = \pi$ if $\phi, \phi' \in \Omega_0 S^1$ have different winding number. (Hint: use the mean value theorem.)

In particular, $\Omega_0 S^1$ is not connected; the sets $\Omega_0^{(k)} S^1$ of loops with winding number k are both open and closed. In fact, one can show that $\Omega_0^{(k)} S^1$ is contractible.

Exercise 7. Show that $\Omega_0^{(k)} S^1$ is homeomorphic (but not isometric) to the set

$$\{\psi\colon [0,1] \to \mathbb{R}\,;\, \psi \text{ is continuous, } \psi(0) = 0, \text{ and } \psi(1) = k\}\,,$$

equipped with the metric $d_\infty(\psi, \psi') = \sup_{t\in[0,1]}|\psi(t) - \psi'(t)|$. Show that this can be contracted to the function $\psi(t) = kt$. In particular, $\Omega_0^{(k)} S^1$ is connected. Show that $\Omega_0^{(k)} S^1$ is homeomorphic to $\Omega_0\mathbb{R}$. (Compare with exercise 4).

We conclude that LS^1 is the disjoint union of connected components

$$LS^1 \simeq \sqcup_{k\in\mathbb{Z}} S^1 \times \Omega_0^k S^1\,,$$

each one homeomorphic to $S^1 \times \Omega_0\mathbb{R}$.

4 Fundamental group

Since LM is a metric space, we can consider continuous paths in LM, and ask whether LM is path connected. A path from $\phi_0 \in LM$ to $\phi_1 \in LM$ is a continuous map $\Phi\colon [0,1] \to LM$ with $\Phi(0) = \phi_0$ and $\Phi(1) = \phi_1$. We already saw that $L\mathbb{R}$ is connected, but LS^1 is not.

The following exercise shows that continuous paths in LM correspond to *homotopies* in M. A homotopy between $\phi_0\colon [0,1] \to M$ and $\phi_1\colon [0,1] \to M$ is a continuous map $F\colon [0,1]\times[0,1] \to M$ such that $F(0,t) = \phi_0(t)$ and $F(1,t) = \phi_1(t)$.

Exercise 8. If $\Phi\colon [0,1] \to LM$ is a path in LM, then we obtain a map $F\colon [0,1]\times [0,1] \to M$ by $F(s,t) = \Phi(s)(t)$. Show that Φ is continuous if and only if F is continuous. (Hint: since $[0,1]$ is compact, Φ and F are continuous if and only if they are uniformly continuous.)

Let $\pi_0(Y)$ be the set of connected components of a topological space Y, and let $\pi_1(Y,y)$ be the *fundamental group* of loops in Y that start and end in y, where loops are identified if there exists a homotopy between them.

Exercise 9. Show that:

a) A path in LM from ϕ_0 to ϕ_1 corresponds to a homotopy with $F(s,0) = F(s,1)$ for all $s \in [0,1]$.

b) A path in $\Omega_x M$ from ϕ_0 to ϕ_1 corresponds to a homotopy with $F(s,0) = F(s,1) = x$ for all $s \in [0,1]$.

c) There is a bijection
$$\pi_0(\Omega_x M) \simeq \pi_1(M,x)\,.$$
In particular, $\Omega_x M$ is connected if and only if the connected component of x in M is simply connected.

Remark 1. This is the very simplest example of a general phenomenon called *transgression*. Roughly speaking, 'higher geometry' of M (in this case $\pi_1(M)$, to be thought of as 'paths modulo surfaces' in M) can be expressed as 'lower geometry' of ΩM (in this case $\pi_0(M)$, to be thought of as 'points modulo paths' in ΩM).

For $M = S^1$, we used the fact that $LS^1 = S^1 \times \Omega_0 S^1$ to see that $\pi_0(LS^1) = \pi_0(S^1) \times \pi_0(\Omega_0 S^1) = \pi_1(S^1)$. It turns out that we can do something similar if M is a *matrix Lie group*.

5 Loop groups and matrix Lie groups

We denote the group of invertible $n \times n$ matrices over $\mathbb{R}$ by $\mathrm{Gl}(n, \mathbb{R})$. A *matrix Lie group* is a closed subgroup G of $\mathrm{Gl}(n, \mathbb{R})$. Every matrix Lie group is a metric space, with a metric inherited from the norm on the ambient space of $n \times n$ matrices. In the remainder of these notes, we focus on *loop groups*, namely loop spaces LG of a matrix group G.

5.1 The matrix Lie group $\mathrm{SO}(n, \mathbb{R})$

We start by giving some examples of matrix Lie groups. The *special orthogonal group* $\mathrm{SO}(n, \mathbb{R})$ is the group of transformations of $\mathbb{R}^n$ that preserve the metric and the orientation,

$$\mathrm{SO}(n, \mathbb{R}) = \{g \in \mathrm{Gl}(n, \mathbb{R})\,;\, \det(g) = 1 \text{ and } (g\vec{v}, g\vec{w}) = (\vec{v}, \vec{w})\ \forall\, \vec{v}, \vec{w} \in \mathbb{R}^n\}\,. \tag{4}$$

Exercise 10. Show that $g \in \mathrm{SO}(n, \mathbb{R})$ if and only if $\det(g) = 1$ and $g^T g = \mathbf{1}$. Conclude that $\mathrm{SO}(n, \mathbb{R})$ is closed in $\mathrm{Gl}(n, \mathbb{R})$, and hence a matrix Lie group.

The following exercise shows that $\mathrm{SO}(2)$ is just a fancy name for the circle group S^1.

Exercise 11. Write $g \in \mathrm{SO}(2)$ as $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, and show that the vectors $\left(\begin{smallmatrix} a \\ c \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} b \\ d \end{smallmatrix}\right)$ in $\mathbb{R}^2$ are orthonormal. Use this to show that every $g \in \mathrm{SO}(2)$ is of the form

$$g(\phi) = \begin{pmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{pmatrix} \tag{5}$$

for some $\phi \in \mathbb{R}$.

The parametrisation $\phi \mapsto g(\phi)$ 'winds the real line around the circle'. Although it is injective only for small values of ϕ, say $\phi \in (-\pi, \pi)$, it is very useful because it yields a homeomorphism $\phi \mapsto g(\phi)$ between the open neighbourhood $U_0 = (-\pi, \pi)$ of 0 in $\mathbb{R}$ and the open neighbourhood $V_{\mathbf{1}} = \mathrm{SO}(2) \backslash \{-\mathbf{1}\}$ of $\mathbf{1}$ in $\mathrm{SO}(2)$.

Exercise 12. The open neighbourhood $V_{\mathbf{1}} \subseteq \mathrm{SO}(2)$ is contractible. Show that this gives rise to a contractible neighbourhood $LV_{\mathbf{1}}$ of $\mathbf{1}$ in $L\mathrm{SO}(2)$, homeomorphic to the open neighbourhood LU_0 of 0 in $L\mathbb{R}$.

Let $g \in \mathrm{SO}(2)$. Since $V_{\mathbf{1}}$ is an open neighbourhood of $\mathbf{1}$, and since left multiplication by g is continuous, the set $g \cdot V_{\mathbf{1}}$ is an open neighbourhood of g in $\mathrm{SO}(2)$.

Exercise 13. Show that every loop $\phi_0 \in L\mathrm{SO}(2)$ has a contractible neighbourhood homeomorphic to LU_0.

This has far-reaching consequences: it means that the space $L\mathrm{SO}(2)$, which is itself topologically quite interesting, can be covered by 'patching together' contractible open subsets of the Banach space $L\mathbb{R}$. We will get back to this point in greater detail later on, after we find good 'parametrisations' around the identity for arbitrary matrix Lie groups.

5.2 The classical Lie groups

Let $\mathrm{Gl}(n,\mathbb{C})$ be the group of all invertible *complex* $n \times n$ matrices. To show that $\mathrm{Gl}(n,\mathbb{C})$ is a matrix Lie group, consider it as a subgroup of $\mathrm{Gl}(2n,\mathbb{R})$ by identifying $X + iY \in \mathrm{Gl}(n,\mathbb{C})$ with the real $2n \times 2n$ matrix $\left(\begin{smallmatrix} X & Y \\ -Y & X \end{smallmatrix}\right) \in \mathrm{Gl}(2n,\mathbb{R})$, where X and Y are real $n \times n$-matrices.

Exercise 14. Show that this is a group homomorphism. Show that it is injective with closed image, and conclude that $\mathrm{Gl}(n,\mathbb{C})$ is a matrix Lie group.

Note that every closed subgroup of a matrix Lie group is again a matrix Lie group. In particular, closed subgroups of $\mathrm{Gl}(n,\mathbb{C})$ are matrix Lie groups.

Exercise 15. The groups of $n \times n$ matrices over $\mathbb{R}$ or $\mathbb{C}$ with determinant 1 are denoted by $\mathrm{Sl}(n,\mathbb{R})$ or $\mathrm{Sl}(n,\mathbb{C})$, respectively. Show that these are matrix Lie groups.

In practice, examples of matrix Lie groups are often derived from bilinear or sesquilinear forms.

Exercise 16. Let B be an $n \times n$ matrix over $\mathbb{R}$, and let $b(\vec{v},\vec{w}) = (\vec{v}, B\vec{w})$ be the corresponding bilinear form. Show that

$$\mathrm{O}(B,\mathbb{R}) = \{g \in \mathrm{Gl}(n,\mathbb{R})\,;\, b(g\vec{v}, g\vec{w}) = b(\vec{v},\vec{w})\ \forall\ \vec{v},\vec{w} \in \mathbb{R}^n\}$$

is a matrix Lie group.

Since intersections of closed groups are closed, we find that $\mathrm{SO}(B,\mathbb{R}) = \mathrm{O}(B,\mathbb{R}) \cap \mathrm{Sl}(n,\mathbb{R})$ is a matrix Lie group. We find back $\mathrm{SO}(n,\mathbb{R})$ as $\mathrm{SO}(B,\mathbb{R})$ with $B = \mathbf{1}$. If $B = \Omega$ is a nondegenerate *skew symmetric* matrix of rank $n = 2m$, then $\mathrm{Sp}(m) = \mathrm{SO}(\Omega,\mathbb{R})$ is called the *symplectic group*.

Exercise 17. Let B be an $n \times n$ matrix over $\mathbb{C}$. Let $b(\vec{v},\vec{w}) = (\vec{v}, B\vec{w})$ be the corresponding bilinear form, and let $h(\vec{v},\vec{w}) = \langle \vec{v}, B\vec{w} \rangle$ be the sesquilinear form. Show that

$$\begin{aligned} \mathrm{O}(B,\mathbb{C}) &= \{g \in \mathrm{Gl}(n,\mathbb{C})\,;\, b(g\vec{v}, g\vec{w}) = b(\vec{v},\vec{w})\ \forall\ \vec{v},\vec{w} \in \mathbb{C}^n\} \quad \text{and} \\ \mathrm{U}(B) &= \{g \in \mathrm{Gl}(n,\mathbb{C})\,;\, h(g\vec{v}, g\vec{w}) = h(\vec{v},\vec{w})\ \forall\ \vec{v},\vec{w} \in \mathbb{C}^n\} \end{aligned}$$

are matrix Lie groups. (Hint: show that $g^T B g = B$ and $g^* B g = B$ are closed expressions.)

It follows from exercise (17) and (15) that $\mathrm{SO}(B,\mathbb{C}) = \mathrm{O}(B,\mathbb{C}) \cap \mathrm{Sl}(n,\mathbb{C})$ and $\mathrm{SU}(B) = \mathrm{U}(B) \cap \mathrm{Sl}(n,\mathbb{C})$ are also matrix Lie groups. If $B = \mathbf{1}$ is the $n \times n$ identity matrix, then we denote $\mathrm{SO}(B,\mathbb{C})$ and $\mathrm{SU}(B)$ by $\mathrm{SO}(n,\mathbb{C})$ and $\mathrm{SU}(n)$, respectively.

We have a closer look at the group $\mathrm{SU}(n)$ of unitary $n \times n$ matrices with determinant 1.

Exercise 18. The group $\mathrm{SU}(n)$ is compact. (Hint: what is the length of the matrix $g \in \mathrm{SU}(n)$ with respect to the hermitean inner product $\langle X, Y \rangle = \mathrm{tr}(\mathrm{X}^*\mathrm{Y})$ on $M_n(\mathbb{C})$?)

Exercise 19. Show that $\mathrm{SU}(2)$ is homeomorphic to S^3, for example in the following way:

a) The orbit map $\mathrm{SU}(2) \mapsto \mathbb{C}^2$ given by $g \mapsto g\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ is continuous and injective.

b) Its image is $S^3 \simeq \{v \in \mathbb{C}^2\,;\, \|v\| = 1\}$.

c) By the previous exercise, the orbit map $\mathrm{SU}(2) \to S^3$ is continuous from a compact space to a Hausdorff space.

5.3 Loop groups as topological groups

The loop space LG of a matrix Lie group G is called a *loop group*.

Exercise 20. Show that the multiplication $(\phi \cdot \phi')(t) = \phi(t)\phi'(t)$ makes LG into a group. What is the identity element? What is the inverse of $\phi \in LG$?

The group $\mathrm{Gl}(n,\mathbb{R})$ is a *topological group*, meaning that the multiplication $\mathrm{Gl}(n,\mathbb{R}) \times \mathrm{Gl}(n,\mathbb{R}) \to \mathrm{Gl}(n,\mathbb{R})\colon (g,h) \mapsto gh$ and the inversion map $\mathrm{Gl}(n,\mathbb{R}) \to \mathrm{Gl}(n,\mathbb{R})\colon g \mapsto g^{-1}$ are continuous. It follows that every matrix Lie group is also a topological group.

Exercise 21. Show that if G is a compact matrix Lie group, then LG is a topological group. (Hint: since G is compact, the multiplication $\mu\colon G \times G \to G$ and inversion $\iota\colon G \to G$ are *uniformly* continuous.)

This result remains valid also for noncompact matrix groups, although the proof is a bit more involved.

Proposition 1. *Every loop group LG is a topological group.*

Proof. We show that multiplication $LG \times LG \to LG$ is continuous. Let $\phi_0, \chi_0 \in LG$. For every $\epsilon > 0$, we need to find a $\delta, \delta' > 0$ such that $d_\infty(\phi, \phi_0) \leq \delta$ and $d_\infty(\chi, \chi_0) \leq \delta'$ imply $d_\infty(\phi\chi, \phi_0\chi_0) \leq \epsilon$.

The idea is to show that everything of interest happens in a compact subset $K(\phi_0) \times K(\chi_0) \subseteq G \times G$. Since multiplication and inversion are uniformly continuous on this compact subset, we can then proceed as in exercise (21).

Let $Z = \{A \in M_n(\mathbb{R})\,;\, \det(A) = 0\}$ be the complement of $\mathrm{Gl}(n,\mathbb{R})$ in $M_n(\mathbb{R})$. Note that $\phi_0(S^1) \subseteq M_n(\mathbb{R})$ is compact as the image of a compact set, and hence closed and bounded. Since $\phi_0(S^1)$ and Z are closed and disjoint, their distance $\Delta = d(\phi_0(S^1), Z)$ is nonzero. Let

$$K(\phi_0) = \{A \in M_n(\mathbb{R})\,;\, d(A, \phi_0(S^1)) \leq \Delta/2\}\,.$$

Note that $K(\phi_0)$ is closed, and it is bounded because $\phi_0(S^1)$ is bounded. It follows that $K(\phi_0) \subseteq M_n(\mathbb{R})$ is compact. Since $d(K(\phi_0), Z) \geq \Delta/2$ by the triangle inequality, we have $K(\phi_0) \cap Z = \emptyset$. It follows that the compact set $K(\phi_0)$ is a subset of $\mathrm{Gl}(n, \mathbb{R})$. If $d_\infty(\phi, \phi_0) \leq \Delta/2$, then for all $t \in [0,1]$, $d(\phi(t), \phi_0(t)) \leq \Delta/2$, and hence $\phi(t) \in K(\phi_0)$.

By choosing $\delta \leq \Delta/2$, we can thus ensure that $\phi\colon [0,1] \to G$ maps into $K(\phi_0)$. Similarly, by choosing $\delta' \leq \Delta'/2$, we can ensure that $\chi_0\colon [0,1] \to G$ maps into a compact neighbourhood $K(\chi_0)$ of $\chi_0(S^1)$ in G.

Since multiplication $\mu\colon G \times G \to G$ is continuous, it is *uniformly* continuous on the compact set $K(\phi_0) \times K(\chi_0) \subseteq G \times G$. In other words, there exist $D, D' > 0$ such that $d(g, g') < D$ and $d(h, h') < D'$ imply $d(gg', hh') < \epsilon$ for all $(g,h), (g', h')$ in $K(\phi_0) \times K(\chi_0)$.

Choose $\delta = \min(\Delta/2, D)$ and $\delta' = \min(\Delta'/2, D')$. If $d_\infty(\phi, \phi_0) \leq \delta$ and $d_\infty(\chi, \chi_0) \leq \delta'$, then one has $d(\phi(t), \phi_0(t)) \leq \delta$ and $d(\chi(t), \chi_0(t)) \leq \delta'$ for all $t \in S^1$. This implies that $\phi(t)$ and $\phi_0(t)$ are in $K(\phi_0)$, and $\chi(t)$ and $\chi_0(t)$ are in $K(\chi_0)$, for all $t \in [0,1]$. By uniform continuity of the multiplication on K_Δ, we thus find $d(\phi(t)\chi_0(t), \phi_0(t)\chi_0(t)) < \epsilon$ for all $t \in S^1$. This means that $d_\infty(\phi\chi, \phi_0\chi_0) < \varepsilon$.

The proof for continuity of the inversion map proceeds in a similar way, using uniform continuity of the inversion on $K(\phi_0)$. □

Exercise 22. Show that LG is homeomorphic to $G \times \Omega_{\mathbf{1}} G$. Conclude that

$$\pi_0(LG) \simeq \pi_0(G) \times \pi_1(G, \mathbf{1}) .$$

(Hint: decompose $\phi \in LG$ into a constant loop and a based loop. To show that this decomposition and its inverse are continuous, use exercises (2) and (3), and continuity of the multiplication in LG.)

6 Lie algebras

In exercise (11), we saw that the map $\mathbb{R} \to \mathrm{SO}(2)$ defined by

$$\phi \mapsto g(\phi) = \begin{pmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{pmatrix}$$

is locally a homeomorphism, so we can use it to parametrise a neighbourhood of the identity in SO(2). In order to generalise this parametrisation to other matrix Lie groups, we reinterpret it as the *exponential map*.

Exercise 23. Calculate $\exp\left(\begin{smallmatrix} 0 & \phi \\ -\phi & 0 \end{smallmatrix}\right)$, where the exponential of a matrix X is given by $\exp(X) = \mathbf{1} + X + \frac{1}{2!}X^2 + \frac{1}{3!}X^3 + \dots$

If we define the linear subspace $\mathfrak{so}(2) \subseteq M_2(\mathbb{R})$ by

$$\mathfrak{so}(2) := \left\{ \begin{pmatrix} 0 & \phi \\ -\phi & 0 \end{pmatrix} ; \phi \in \mathbb{R} \right\} ,$$

then we can identify the parametrisation $\phi \mapsto g(\phi)$ of equation (5) with the exponential map $\exp\colon \mathfrak{so}(2) \to \mathrm{SO}(2)$.

6.1 The Lie algebra $\mathfrak{so}(n, \mathbb{R})$

This is the picture that carries over to $\mathrm{SO}(n, \mathbb{R})$. In the following exercise, we will see that there exists a biggest linear subspace $\mathfrak{so}(n, \mathbb{R}) \subseteq M_n(\mathbb{R})$, called the *Lie algebra* of $\mathrm{SO}(n, \mathbb{R})$, such that the exponential map $\exp\colon M_n(\mathbb{R}) \to \mathrm{Gl}(n, \mathbb{R})$ maps $\mathfrak{so}(n, \mathbb{R})$ into $\mathrm{SO}(n, \mathbb{R})$.

Exercise 24. Let $\mathfrak{so}(n, \mathbb{R}) \subseteq M_n(\mathbb{R})$ be the biggest linear subspace of $M_n(\mathbb{R})$ such that $\exp(\mathfrak{so}(n, \mathbb{R})) \subseteq \mathrm{SO}(n, \mathbb{R})$.

a) If $X \in \mathfrak{so}(n, \mathbb{R})$, then $X^T + X = 0$. (Hint: we know that $\exp(tX) \in \mathrm{SO}(n, \mathbb{R})$ for all $t \in \mathbb{R}$. Now differentiate at $t = 0$.)

b) If $X^T + X = 0$, then $\exp(X) \in \mathrm{SO}(n, \mathbb{R})$. (Hint: to show that $\exp(X)$ is orthogonal, one can use that $\exp(X)^T = \exp(X^T)$. For the determinant condition, note that $\det(g) = \pm 1$ for every orthogonal matrix g. Now use the fact that $t \mapsto \det(\exp(tX))$ is continuous.)

c) Show that $\mathfrak{so}(n, \mathbb{R}) = \{X \in M_n(\mathbb{R})\,;\, X^T + X = 0\}$ is a linear subspace of $M_n(\mathbb{R})$, and conclude that it is the Lie algebra of $\mathrm{SO}(n, \mathbb{R})$.

Later on, we will see that exp gives a homeomorphism between an open neighbourhood U_0 of 0 in $\mathfrak{so}(n, \mathbb{R})$ and an open neighbourhood $V_{\mathbf{1}}$ of $\mathbf{1}$ in $\mathrm{SO}(n, \mathbb{R})$. This means that we can parametrise an open neighbourhood of $\mathrm{SO}(n, \mathbb{R})$ with an open neighbourhood of the *vector space* $\mathfrak{so}(n, \mathbb{R})$.

Exercise 25. What is the dimension d of $\mathfrak{so}(n, \mathbb{R})$? This is the number of parameters needed to parameterise a neighbourhood of $\mathrm{SO}(n, \mathbb{R})$.

Exercise 26. If $X, Y \in \mathfrak{so}(n, \mathbb{R})$, then the commutator $[X, Y] = XY - YX$ is again an element of $\mathfrak{so}(n, \mathbb{R})$.

6.2 The Lie algebras $\mathfrak{gl}(n, \mathbb{C})$ and $\mathfrak{su}(n)$

Exercise 27. Let $\mathfrak{gl}(n, \mathbb{C}) \subseteq M_{2n}(\mathbb{R})$ be the Lie algebra of $\mathrm{Gl}(n, \mathbb{C})$, namely the biggest linear subspace such that $\exp(\mathfrak{gl}(n, \mathbb{C})) \subseteq \mathrm{Gl}(n, \mathbb{C})$.

a) Show that $\mathfrak{gl}(n, \mathbb{C}) = \{\left(\begin{smallmatrix} X & Y \\ -Y & X \end{smallmatrix}\right)\,;\, X, Y \in M_n(\mathbb{R})\}$.

b) Check that the natural inclusion of $M_n(\mathbb{C})$ in $M_{2n}(\mathbb{R})$ by $X + iY \mapsto \left(\begin{smallmatrix} X & Y \\ -Y & X \end{smallmatrix}\right)$ is an algebra homomorphism. Conclude that the commutator bracket and exponential map on $M_n(\mathbb{C})$ agree with those on $M_{2n}(\mathbb{R})$.

This allows us to consider $\mathfrak{gl}(n, \mathbb{C}) \simeq M_n(\mathbb{C})$, with $\exp\colon \mathfrak{gl}(n, \mathbb{C}) \to \mathrm{Gl}(n, \mathbb{C})$ the complex exponential.

Exercise 28. Let $\mathfrak{su}(n)$ be the Lie algebra of $\mathrm{SU}(n)$, namely the biggest linear subspace of $M_n(\mathbb{C})$ such that $\exp(\mathfrak{su}(n)) \subseteq \mathrm{SU}(n)$. Show that

$$\mathfrak{su}(n) = \{X \in M_n(\mathbb{C})\,;\, \mathrm{tr}(X) = 0, \text{ and } X^* + X = 0\}\,.$$

(Hint: recall that $g \in \mathrm{SU}(n)$ if $\det(g) = 1$ and $g^* g = \mathbf{1}$. Now have another look at Exercise 24. For the trace condition, note that $X^* + X = 0$ implies that X is diagonalisable.)

Exercise 29. What are the Lie algebras of $\mathrm{O}(B, \mathbb{C})$ and $\mathrm{U}(B)$? And for $\mathrm{SO}(B, \mathbb{C})$ and $\mathrm{SU}(B)$?

6.3 Lie algebras of matrix Lie groups

Definition 1. The *Lie algebra* $\mathrm{Lie}(G) \subseteq M_n(\mathbb{R})$ of a matrix Lie group $G \subseteq M_n(\mathbb{R})$ is the biggest linear subspace of $M_n(\mathbb{R})$ such that $\exp(\mathrm{Lie}(G)) \subseteq G$.

We will see that the vector space $\mathrm{Lie}(G)$ is closed closed under the commutator bracket $[X, Y] = XY - YX$, which explains the 'algebra' in 'Lie algebra'. In the process, we will prove the following convenient characterisation:

$$\mathrm{Lie}(G) = \{X \in M_n(\mathbb{R})\,;\, \exp(tX) \in G\, \forall\, t \in \mathbb{R}\}\,. \tag{6}$$

Finally, we will show that the exponential map yields a homeomorphism between an open neighbourhood of 0 in the vector space $\mathrm{Lie}(G)$ and an open neighbourhood of $\mathbf{1}$ in G.

Lemma 2. *For $X, Y \in M_n(\mathbb{R})$, we have*

$$\exp(X+Y) = \lim_{k\to\infty} \big(\exp(X/k)\exp(Y/k)\big)^k \tag{7}$$

$$\exp([X,Y]) = \lim_{k\to\infty} \Big(\exp(X/k)\exp(Y/k)\exp(-X/k)\exp(-Y/k)\Big)^{k^2}. \tag{8}$$

Proof. The expression in the brackets of (7) and (8) can be expanded as

$$\exp(X/k)\exp(Y/k) = \mathbf{1} + (X+Y)/k + \mathcal{O}(k^{-2})\,,$$
$$\exp(X/k)\exp(Y/k)\exp(-X/k)\exp(-Y/k) = \mathbf{1} + [X,Y]/k^2 + \mathcal{O}(k^{-3})\,.$$

The above formulæ then follow from the fact that $\lim_{k\to\infty}(1 + A/k + B(k))^k = \exp(A)$ for every matrix $A \in M_n(\mathbb{R})$, and for every sequence $B(k)$ of matrices with $\lim_{k\to\infty} k\|B(k)\| = 0$. □

Theorem 3. *The Lie algebra of a matrix group G is given by*

$$\mathrm{Lie}(G) = \{X \in M_n(\mathbb{R})\,;\, \exp(tX) \in G\, \forall\, t \in \mathbb{R}\}\,. \tag{9}$$

It is closed under the commutator bracket; if $X, Y \in \mathrm{Lie}(G)$, then $[X, Y] = XY - YX$ is also in $\mathrm{Lie}(G)$.

Proof. Temporarily denote the r.h.s. of (9) by $\mathrm{lie}(G)$. Clearly, every linear subspace $V \subseteq M_n(\mathbb{R})$ with $\exp(V) \subseteq G$ satisfies $V \subseteq \mathrm{lie}(G)$. To show that $\mathrm{lie}(G)$ is the Lie algebra of G, it therefore suffices to show that it is a vector space.

By definition, $\mathrm{lie}(G)$ is closed under scalar multiplication. To show that it is closed under addition, suppose that $X, Y \in \mathrm{lie}(G)$. Then $\exp(X/k)$ and $\exp(Y/k)$ are in G by definition. Since G is a group, $(\exp(X/k)\exp(Y/k))^k \in G$, and since G is closed, (7) shows that $\exp(X+Y)$ is in G. Since the same holds for all scalar multiples of X and Y, this implies that $X+Y \in \mathrm{lie}(G)$. In the same way, one uses equation (8) to show that $\mathrm{lie}(G)$ is closed under the commutator bracket, $[X, Y] \in \mathrm{lie}(G)$ for all $X, Y \in \mathrm{lie}(G)$. □

Theorem 4 (Von Neumann)**.** *There exists a neighbourhood $U_0 \subseteq \mathrm{Lie}(G)$ of zero such that $\exp(U_0) \subseteq G$ is a neighbourhood of $\mathbf{1}$ in G, and the exponential map defines a homeomorphism $\exp\colon U_0 \xrightarrow{\sim} \exp(U_0)$.*

Proof. We show that $\exp(\mathrm{Lie}(G)) \subseteq G$ is a neighbourhood of $\mathbf{1}$. Define the inner product $(A, B) = \mathrm{tr}(A^T B)$ on $M_n(\mathbb{R})$, and let $\mathrm{Lie}(G)^\perp$ be the orthogonal complement of $\mathrm{Lie}(G)$ in $M_n(\mathbb{R})$. Let $E\colon M_n(\mathbb{R}) \to M_n(\mathbb{R})$ be the smooth map defined by $E(X+Y) = \exp(X)\exp(Y)$ for $X \in \mathrm{Lie}(G)$ and $Y \in \mathrm{Lie}(G)^\perp$. Since its derivative at 0 is the identity, it is a local diffeomorphism around 0 by the inverse function theorem.

Suppose that $\mathbf{1}$ is not in the interior of $\exp(\mathrm{Lie}(G)) \subseteq G$. Then there exists a sequence g_k in $G \setminus \exp(\mathrm{Lie}(G))$ with $\lim_{k\to\infty} g_k = \mathbf{1}$. For k sufficiently large, we can write $g_k = \exp(X_k)\exp(Y_k)$ with $X_k \in \mathrm{Lie}(G)$, $Y_k \in \mathrm{Lie}(G)^\perp$, $X_k, Y_k \to 0$, and $Y_k \neq 0$. As g_k and $\exp(X_k)$ are in G, we also have $\exp(Y_k) \in G$. Passing to a subsequence if necessary, we use compactness of the unit sphere in $\mathrm{Lie}(G)^\perp$ to show that $Y_k/\|Y_k\| \to Y$ for some $Y \in \mathrm{Lie}(G)^\perp$ of norm one. For any $t \in \mathbb{R}$, choose integers n_k such that $n_k\|Y_k\| \to t$. Then $\exp(n_k Y_k) \to \exp(tY)$, and $\exp(tY)$ lies in G by continuity. Since this holds for every $t \in \mathbb{R}$, we conclude that $Y \in \mathrm{Lie}(G)$, contradicting the assumption. □

7 Matrix Lie groups as manifolds

Definition 2. A Hausdorff topological space M is called a *topological manifold* of dimension d if every point x has a neighbourhood V_x homeomorphic to an open subset $U_x \subseteq \mathbb{R}^d$. A homeomorphism $\kappa_x\colon V_x \to U_x$ is called a *chart*, and V_x is called a *coordinate neighbourhood* of x.

Corollary 5. *Every matrix Lie group G is a topological manifold of dimension $d = \dim(\mathrm{Lie}(G))$.*

Proof. We cover G by open neighbourhoods $V_g = g\exp(U_0)$, where g runs over G and $U_0 \subseteq \mathrm{Lie}(G) \simeq \mathbb{R}^d$ is as in Theorem 4. The chart $\kappa\colon V_g \to U_0$, given by the inverse of $\kappa^{-1}(X) = g\exp(X)$, is a homeomorphism, since both the exponential map and left multiplication by g are (locally) homeomorphisms. □

If M is a topological manifold and $V_x, V_y \subseteq M$ are two open subsets with nonzero intersection, then the *transition function*

$$\kappa_y \circ \kappa_x^{-1}\colon U_x \cap \kappa_x^{-1}(V_y) \to \kappa_y^{-1}(V_x) \cap U_y$$

is a homeomorphism. Since the transition functions $\kappa_x^{-1} \circ \kappa_y$ are maps between open subsets of $\mathbb{R}^d$, it makes sense to ask whether or not they are *smooth.* A topological manifold is called a *smooth manifold* if this is the case.

Definition 3. A smooth manifold is a topological manifold, equipped with distinguished charts $\kappa_x\colon U_x \to V_x$ such that all transition functions are smooth.

Corollary 6. *Every matrix Lie group is a smooth manifold.*

Proof. Since the map $\exp\colon M_n(\mathbb{R}) \to M_n(\mathbb{R})$ has $D\exp|_0 = \mathbf{1}$, it is local diffeomorphism between an open neighbourhood U_0 of 0 and an open neighbourhood $V_{\mathbf{1}} = \exp(U)$ of $\mathbf{1}$ by the inverse function theorem. Denote its local inverse by $\log\colon V_{\mathbf{1}} \to U_0$. If $U_g \cap U_h$ is nonempty, and $\kappa_g(X) = \kappa_h(Y)$, then $g\exp(X) = h\exp(Y)$ yields $Y = \log(h^{-1}g\exp(Y)) = \kappa_h^{-1} \circ \kappa(g)(X)$. Since the logarithm and left multiplication by $h^{-1}g$ are both smooth, this is a smooth transition function. $\square$

If M and M' are smooth manifolds, and $f\colon M \to M'$ is a continuous map with $f(x) = y$, then there exist open coordinate neighbourhoods $V_x \subseteq M$ of x and $V_y \subseteq M'$ of y such that f restricts to a map $f\colon V_x \to V'_y$. Using the charts $\kappa_x\colon V_x \to U_x$ and $\kappa'_y\colon V'_y \to U'_y$, we identify V_x with $U_x \subseteq \mathbb{R}^d$ and V'_y with $U'_y \subseteq \mathbb{R}^{d'}$. We thus obtain a continuous map

$$\tilde{f}\colon U_x \to U_y, \quad \tilde{f} = \kappa'_y \circ f \circ \kappa_x^{-1}\,.$$

Since this is a map from an open subset of $\mathbb{R}^d$ to an open subset of $\mathbb{R}^{d'}$, it makes sense to ask whether it is smooth or not.

Definition 4. A continuous map $f\colon M \to M'$ is called *smooth* if for every $x \in M$, there exist coordinate neighbourhoods such that $\tilde{f}\colon U_x \to U_y$ is smooth.

Exercise 30. Check that smoothness of the transition maps implies that the above condition is independent of the choice of charts.

Exercise 31. If M and M' are smooth manifolds, then the product $M \times M'$ is a smooth manifold.

Definition 5. A *Lie group* is a group G with the structure of a smooth manifold, such that multiplication $\mu\colon G \times G \to G$ and inversion $\iota\colon G \to G$ are smooth.

Exercise 32. In view of Corollary 6, every matrix Lie group G is a smooth manifold. Prove that multiplication and inversion are smooth maps.

Corollary 7. *Every matrix Lie group is a Lie group.*

8 Loop groups as topological Banach manifolds

A topological manifold is a topological space that is locally homeomorphic to the finite dimensional vector space $\mathbb{R}^N$. Clearly, expecting $\phi_0 \in LG$ to have an open neighbourhood homeomorphic to an open neighbourhood in $\mathbb{R}^N$ is too much to ask for. Something else is true though: every $\phi_0 \in LG$ has an open neighbourhood homeomorphic to an open neighbourhood in $L\mathbb{R}^d$, where d is the dimension of the Lie algebra $\mathrm{Lie}(G)$.

Proposition 8. *Every $\phi_0 \in LG$ has a neighbourhood $\phi_0 \cdot LV_{\mathbf{1}}$ that is homeomorphic to $LU_0 \subseteq L\mathrm{Lie}(G)$, where $U_0 \subseteq \mathrm{Lie}(G)$ is an open neighbourhood of* 0. *The homeomorphism*

$$\kappa_{\phi_0}\colon \mathrm{Lie}(G) \supseteq LU_0 \to \phi_0 \cdot LV_{\mathbf{1}} \subset LG$$

is given by $\kappa_{\phi_0}(\xi)(t) = \phi_0(t)\exp(\xi(t))$.

Proof. Since $U_0 \subseteq \mathrm{Lie}(G)$ is homeomorphic to $V_{\mathbf{1}} \subseteq G$, the set $LU_0 \subseteq L\mathrm{Lie}(G)$ is homeomorphic to $LV_{\mathbf{1}} \subseteq LG$. We show that $LV_{\mathbf{1}}$ is an open neighbourhood of $\mathbf{1}$ in LG.

Since $V_{\mathbf{1}}$ is an open neighbourhood of $\mathbf{1}$, there exists a $\delta > 0$ such that $d(g, \mathbf{1}) < \delta$ implies $g \in V_{\mathbf{1}}$. Now if $d_\infty(\phi, \mathbf{1}) < \delta$, then $d(\phi(t), \mathbf{1}) < \delta$, hence $\phi(t) \in V_{\mathbf{1}}$, for all $t \in [0, 1]$. Thus $d_\infty(\phi, \mathbf{1}) < \delta$ implies $\phi \in LV_{\mathbf{1}}$. The proof that LU_0 is an open neighbourhood of 0 in $\mathrm{Lie}(G)$ is similar. Since LG is a topological group, the map $\phi \mapsto \phi_0 \cdot \phi$ is a homeomorphism $LG \to LG$. It follows that $\phi_0 LV_{\mathbf{1}}$ is an open neighbourhood of $\phi_0 \in LG$. □

The space $L(\mathrm{Lie}(G)) \simeq L\mathbb{R}^d$ is a *Banach space.* If you can't have finite dimensional vector spaces, then Banach spaces are the next best thing.

Definition 6. A Banach space F is a vector space with a norm $\| \cdot \| \colon F \to \mathbb{R}^{\geq 0}$ such that F is complete; every Cauchy sequence f_n converges in F.

Exercise 33. Show that $L\mathbb{R}^d$, equipped with the norm $\|\phi\|_\infty = \sup_{t\in[0,1]} \|\phi(t)\|$, is a Banach space.

(Step 1: prove that for every t, $\lim_{k\to\infty} \phi_k(t) = \phi(t)$ exists. Step 2: prove that $\|\phi - \phi_k\|_\infty \to 0$. For this, write $\phi - \phi_n = (\phi - \phi_m) + (\phi_m - \phi_n)$. Bound the second term by $\epsilon/2$ for $m, n > N$. Write the first term as $\lim_{k\to\infty} \phi_k - \phi_m$ and bound by $\epsilon/2$ as well. Step 3: show that ϕ is continuous by an $\epsilon/3$-argument with $\phi(t)$, $\phi(t')$, $\phi_n(t)$ and $\phi_n(t')$. Use that ϕ_n is uniformly continuous on the compact set $[0, 1]$.)

Since a Banach space is the next best thing to a finite dimensional vector space, a *Banach topological manifold* is the next best thing to a topological manifold.

Definition 7. A *Banach topological manifold* $\mathcal{M}$ is a topological Hausdorff space that is locally homeomorphic to a Banach space F; every point $\phi \in \mathcal{M}$ has an open neighbourhood $\mathcal{V}_\phi$ that is homeomorphic to a neighbourhood $\mathcal{U}_\phi$ in F.

Again, these open neighbourhoods $\mathcal{V}_\phi \subseteq \mathcal{M}$ are called *coordinate neighbourhoods*, and the homeomorphisms $\kappa_\phi \colon \mathcal{V}_\phi \to \mathcal{U}_\phi$ are called *charts.* By Proposition 8, every point $\phi_0 \in LG$ has a neighbourhood homeomorphic to the open neighbourhood LU in the Banach space $L(\mathrm{Lie}(G)) \simeq L\mathbb{R}^d$. It follows that LG is a topological Banach manifold.

Corollary 9. *For any matrix Lie group G, the topological group LG is a topological Banach manifold.*

9 Loop groups as Banach Lie groups

A *smooth* manifold is a topological manifold such that the transition maps $U \to U'$ between the open subsets of $\mathbb{R}^d$ are smooth. To define *smooth* Banach manifolds, we therefore need to define what *smooth* maps between (open subsets of) Banach spaces are. There are several ways to do this. The following is based on the notion of *partial* derivatives.

Definition 8. Let F, F' be Banach spaces, and let $U \subseteq F$ be an open neighbourhood. A continuous map $T \colon U \to F'$ is called *differentiable* at $f \in U$ if there exists a (necessarily unique) bounded linear map $D_f T \colon F \to F'$ such that

$$DT_f(h) := \lim_{h\to 0} \frac{T(f + h) - T(f)}{\epsilon} .$$

It is C^1 if the *derivative* $DT\colon U \times F \to F'$ is continuous. The map T is C^n if DT is C^{n-1}, and it is *smooth* if it is C^n for all $n \in \mathbb{N}$.

Exercise 34. Check that for $F = \mathbb{R}^n$ and $F' = \mathbb{R}^m$, this gives back the usual notion of smooth maps.

We have a closer look at maps of the form

$$T_\tau\colon L\mathbb{R}^n \to L\mathbb{R}^m, \qquad T_\tau(f)(t) = \tau_t(f(t))\,, \tag{10}$$

where $\tau\colon [0,1] \times \mathbb{R}^n \to \mathbb{R}^m$ is a periodic map; $\tau(0,x) = \tau(1,x)$ for all $x \in \mathbb{R}^n$.

Exercise 35. Suppose that for all $t \in [0,1]$, the map $\tau_t\colon \mathbb{R}^n \to \mathbb{R}^m$ is C^1, and suppose that the first derivatives of τ_t in the x-direction are continuous in t as well as x. Let $U \subseteq \mathbb{R}^n$ be a bounded, open set. Show that on $[0,1] \times U$, the derivatives of τ_t in the x-direction are bounded by a constant K_1 which does not depend on t or x. Show that the map T_τ is continuous.

Exercise 36. In the situation of exercise (35), suppose that for each $t \in [0,1]$, the map $\tau_t\colon \mathbb{R}^n \to \mathbb{R}^m$ is C^2, with first and second derivatives in the x-direction that are continuous in t as well as x. Show that on the bounded open set $[0,1] \times U$, the second derivatives of τ_t in the x-direction are bounded by a constant K_2 which does not depend on t or x.

a) For $f, h \in L\mathbb{R}^n$, choose a bounded open set $U \subseteq \mathbb{R}^n$ that contains the image of $f + \epsilon h$ for all $0 \leq \epsilon \leq 1$. Use the taylor series to show that for every $t \in [0,1]$,

$$\frac{\tau_t(f(t) + \epsilon h(t)) - \tau_t(f(t))}{\epsilon} = D|_{f(t)}\tau_t \cdot h(t) + \epsilon R(t)\,,$$

with remainder $|R(t)| \leq K_2 h^2(t)$. Conclude that $DT_f(h) \in L\mathbb{R}^m$ is the loop given by $t \mapsto (D|_{f(t)}\tau_t) \cdot h(t)$.

b) Show that $DT\colon L\mathbb{R}^n \times L\mathbb{R}^n \to L\mathbb{R}^m$ is of the form $T_{\tilde\tau}$ for the continuous map

$$\tilde\tau\colon [0,1] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m\,, \quad \tilde\tau_t(x,y) = D|_x\tau_t \cdot y\,.$$

Show that the first derivatives of $\tilde\tau_t(x,y)$ in the x and y-direction are continuous. Use exercise (35) to conclude that DT is continuous, and hence that T is C^1.

Exercise 37. Suppose that for every $t \in [0,1]$, the map $\tau_t\colon \mathbb{R}^n \to \mathbb{R}^m$ is smooth. Suppose that all partial derivatives are continuous in t as well as x. Show that the map $T_\tau\colon L\mathbb{R}^n \to L\mathbb{R}^m$ of equation (10) is smooth.

Exercise 38. Check that the transition functions for the topological Banach manifold LG are of the type discussed in exercise (37). Conclude that LG is a smooth Banach manifold.

The definition of *smooth maps* between smooth Banach manifolds is completely analogous to the case of finite dimensional smooth manifolds. A continuous map $T\colon \mathcal{M} \to \mathcal{M}'$ is called *smooth* if for every $\phi \in \mathcal{M}$, there exist open coordinate neighbourhoods $\mathcal{V}_\phi \subseteq \mathcal{M}$ of ϕ and $\mathcal{V}'_{T(\phi)} \subseteq \mathcal{M}'$ of $T(\phi)$ such that

$$\widetilde{T}\colon \mathcal{U}_\phi \to \mathcal{U}'_{T(\phi)}\,, \quad \widetilde{T} = \kappa'_{T(\phi)} \circ T \circ \kappa_\phi^{-1}$$

is smooth.

Exercise 39. Show that the concatenation of smooth maps between Banach spaces is smooth. Conclude that the above definition is independent of the choice of charts.

A *Banach Lie group* is a group $\mathcal{G}$ with the structure of a smooth Banach manifold, such that multiplication $\mu\colon \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ and inversion $\iota\colon \mathcal{G} \to \mathcal{G}$ are smooth.

Theorem 10. *The loop group LG is a Banach Lie group.*

Proof. identify the neighbourhood $\phi \cdot LV_{\mathbf{1}}$ of $\phi \in LG$ with the neighbourhood LU_0 of $0 \in L(\mathrm{Lie}(G))$ by the inverse chart $LU_0 \ni X \mapsto \phi_t \exp(X) \in \phi \cdot LV_{\mathbf{1}}$. To show smoothness of the inversion $\iota\colon \phi \cdot LV_{\mathbf{1}} \to \phi^{-1} \cdot LV'_{\mathbf{1}}$, identify it with the map $LU'_0 \to LU_0$ given by

$$X_t \mapsto \log(\phi_t \exp(-X_t)\phi_t^{-1})$$

and apply exercise (37). (If necessary, choose U'_0 sufficiently small for the image of LU'_0 to land in LU_0.) To show smoothness of the multiplication, note that the open set $\phi \cdot LV_{\mathbf{1}} \times \psi \cdot LV_{\mathbf{1}}$ is a neighbourhood of (ϕ, ψ) in the smooth Banach manifold $LG \times LG \simeq L(G \times G)$. Using the above charts, identify the multiplication $\mu\colon \phi \cdot LV'_{\mathbf{1}} \times \psi \cdot LV'_{\mathbf{1}} \to \phi\psi \cdot LV_{\mathbf{1}}$ with

$$LU'_0 \times LU'_0 \to LU_0\,, \quad (X_t, Y_t) \mapsto \log(\psi_t^{-1} \exp(X_t)\psi_t \exp(Y_t))$$

in the coordinate charts. Again, apply the result of exercise (37). □

10 Outlook

The main advantage of smooth manifolds over topological spaces is that the machinery of differential calculus becomes available. This can be used, for example, to compute extrema of smooth functions $f\colon M \to \mathbb{R}$, or to do Morse theory.

Large parts of this powerful machinery carry over to smooth Banach manifolds, making them much more amenable to analysis than other topological spaces. For example, an elegant proof of the fact that $\pi_2(G) = \{0\}$ for compact Lie groups G can be given by translating this to $\pi_1(\Omega_{\mathbf{1}}G) = \{0\}$, which in turn can be proven by constructing a suitable cell decomposition of the loop space [PS86, Corr. 8.6.7].

There are other notions of 'infinite dimensional manifolds' which support a form of differential calculus. Particularly useful is the concept of *Fréchet manifolds*, which are locally homeomorphic to Fréchet spaces. For example, the smooth loop group $C^\infty(S^1, G)$ is a Fréchet Lie group, meaning that it is a Fréchet manifold with smooth multiplication and inversion. Although this is just a variation on the continuous loop group LG, it has a radically different (and much richer) representation theory.

There are also Fréchet Lie groups which do not have 'Banach siblings'. For example, if M is a compact smooth manifold, then the group $\mathrm{Diff}(M)$ of diffeomorphisms is a Fréchet Lie group that locally looks like $\mathrm{Vec}(M)$, the Lie algebra of smooth vector fields on M.

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